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VISUAL AIDS IN MODERN ALGEBRA

BY T. DONNELLAN

The purpose of these notes is to suggest diagrams that may illustrate the setting up and ordering of equivalence classes, their addition and multiplication.

(1) *Integers*

Denoting the natural numbers by $a, b, c, d, \ldots$, we classify all ordered pairs (a, b) by the equivalence relation $(a, b) = (c, d)$ if and only if $a+d = b+c$. We define the sum of the two classes containing (a, b), (c, d) respectively as the class containing $(a+c, b+d)$, and their product as the class containing $(ac+bd, ad+bc)$. The classes thus related form an integral domain and are called the integers. If $a > b$, we denote the class containing (a, b) by $+(a-b)$

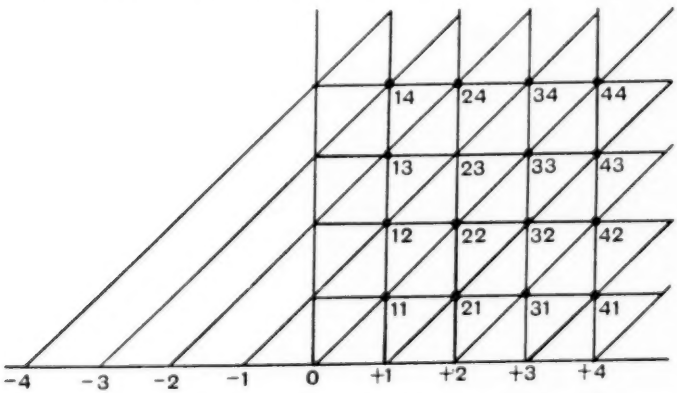

FIG. 1. Integers: Network and ordered pencil.

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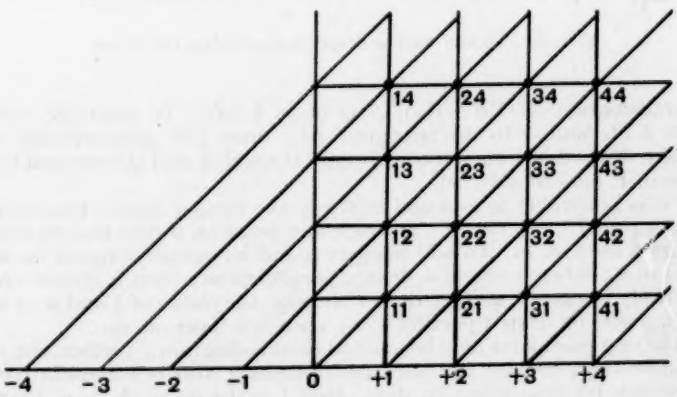

FIG. 1. Integers: Network and ordered pencil.

and call it a positive integer; if $a<b$, we denote the class by $-(b-a)$ and call it a negative integer; if $a=b$, we denote the class indifferently by ± 0 and call it zero. Since the positive integers are isomorphic with the natural numbers considered as an additive or multiplicative semi-group, we frequently omit the + sign.

Label the intersections of a square network systematically with ordered pairs of natural numbers as in Fig. 1. The members of the equivalence classes lie on the parallel diagonals, and these parallels may be taken to represent the integers. The expected order of the integers is displayed by the equally spaced intersections with any convenient transversal.

Fig. 2 shows how we may add and multiply, using representative point members of the integer classes. Let X be (a, b), Y (c, d). To add, complete

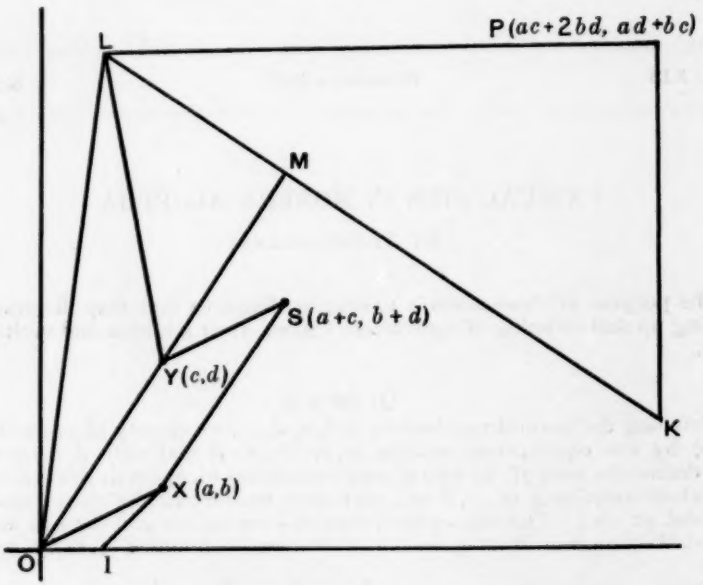

FIG. 2. To add and multiply integer class members.

the parallelogram $XOYS$; then S is $(a+c, b+d)$. To multiply, take the triangle LYO similar to the triangle $X1O$; draw LM perpendicular to OY and make $MN=LM$; then the horizontal through L and the vertical through N meet in P $(ac+bd, ad+bc)$.

But it is preferable to add and multiply the integer classes themselves, as lines of a pencil. See Fig. 3. Through any point on 0 (the line representing zero) draw lines m, n. To add integers u and v: where u meets m, draw p parallel to n; where v meets n, draw q parallel to m; then p, q meet on $u+v$. To multiply integers x and y: draw r through the meets of 1 and m, x and n; where y meets m, draw s parallel to r; then s, n meet on xy.

All the ring postulates may be verified on this diagram; further, the pecked lines show that, besides addition, multiplication also is commutative; the construction for multiplication shows that 1 is the unity element, that there are no divisors of zero and that, with the exception of 1 and -1, no integer

possesses a multiplicative inverse. Hence the integers form an integral domain, but not a field.

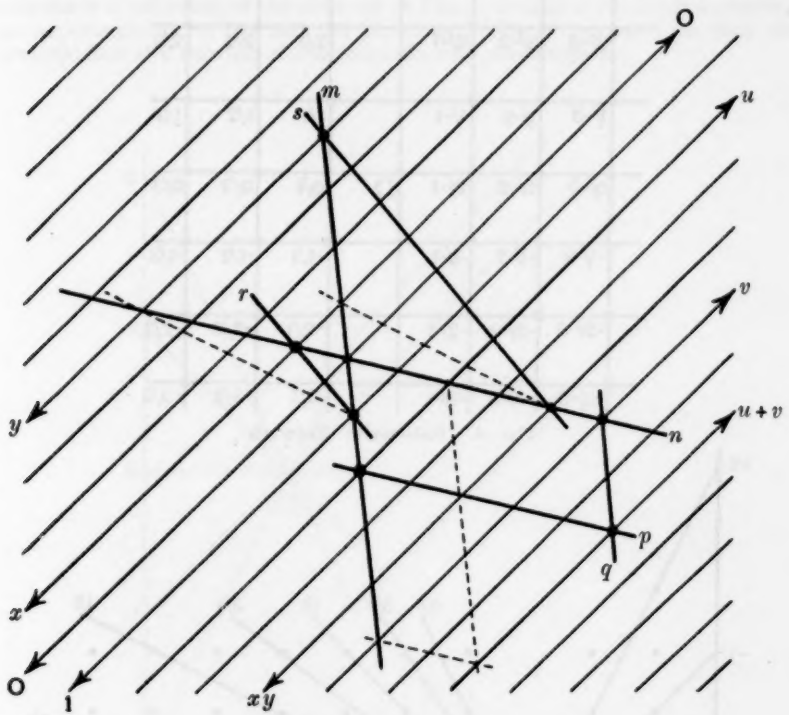

FIG. 3. To add and multiply integers.

(2) *Rationals*

Let a, b, c, d now denote integers. We classify all ordered pairs (a, b), $b \neq 0$, by the equivalence relation $(a, b) = (c, d)$ if and only if $ad = bc$. We define the sum of the two classes containing (a, b), (c, d) respectively as the class containing $(ad + bc, bd)$, and their product as the class containing (ac, bd). The classes thus related form a commutative field. The ordered pair (a, b) is written a/b and called a fraction, positive, zero or negative with ab. The class of all fractions equivalent to a/b is called a rational, positive, zero or negative with a/b. We denote the rational by a/b or any equivalent. Since rationals such as ab/b form a subring isomorphic with the ring of integers, we usually replace ab/b by its integer isomorph a.

Label the intersections of a square network systematically with ordered pairs of integers as in Fig. 4, leaving the vertical through the origin Ω blank. Each labelled intersection represents a fraction; the equivalent fractions lie on the pencil of lines joining every labelled intersection to the origin Ω. The rays of this pencil—some are shown in Fig. 5—may be taken to represent the rationals. The intersections of the rays of the pencil with any convenient parallel to the unlabelled line exhibit the rationals in their expected

3/−3	3/−2	3/−1		3/1	3/2	3/3
2/−3	2/−2	2/−1		2/1	2/2	2/3
1/−3	1/−2	1/−1		1/1	1/2	1/3
0/−3	0/−2	0/−1	Ω	0/1	0/2	0/3
−1/−3	−1/−2	−1/−1		−1/1	−1/2	−1/3
−2/−3	−2/−2	−2/−1		−2/1	−2/2	−2/3
−3/−3	−3/−2	−3/−1		−3/1	−3/2	−3/3

FIG. 4. Rationals: Network.

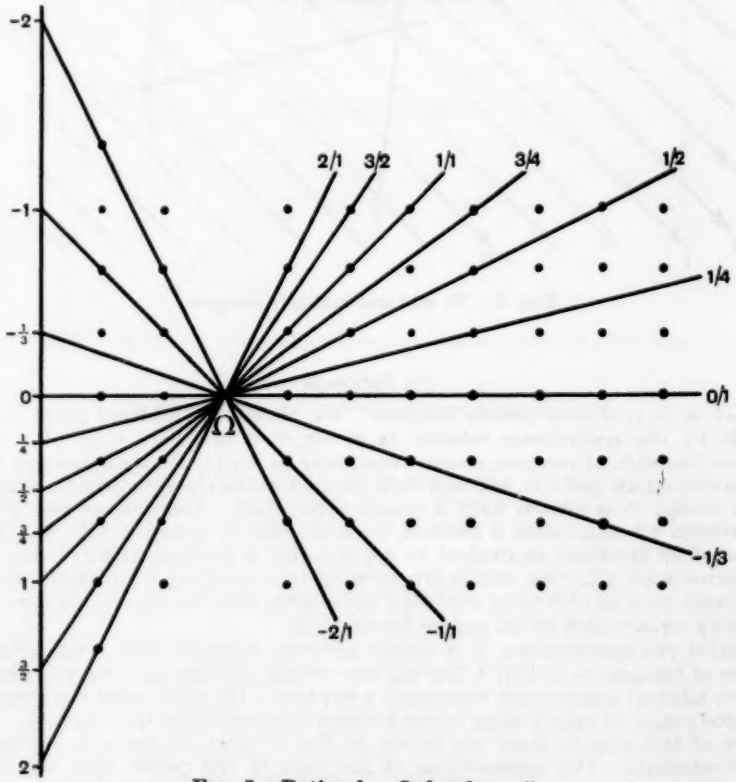

FIG. 5. Rationals: Ordered pencil.

order and correctly spaced. The diagram shows that the rationals form a dense set and suggests that not all points of a straight line can be set in one-to-one correspondence with the rationals. Further, the first quadrant of the network is a reflection of the network of Fig. 1, so that if the integer parallels are superimposed on the rational pencil as in Fig. 6, we have on view the isomorphism of a subring of the rationals with the integers.

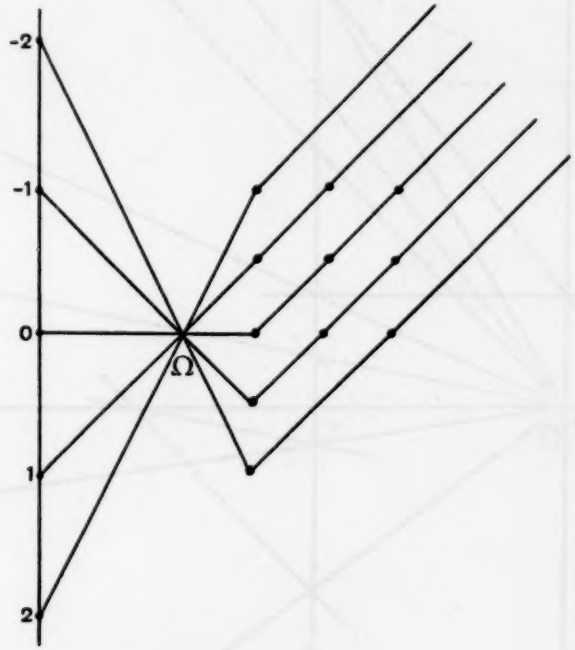

FIG. 6. Isomorphism $n \sim n/1$.

Constructions for addition and multiplication using representative fractions, that is, point members of the rational classes, may easily be devised; but it is preferable to add and multiply the rationals themselves, that is, the rays of the pencil through Ω. See Fig. 7. Calling the unlabelled line ω, draw any line p parallel to ω. To add rationals u and v: where u meets p, draw m parallel to 0/1; where m meets ω, draw n parallel to v; then p, n meet on $u+v$. To multiply rationals x and y: where x meets p, draw r parallel to 1/1; where r meets 0/1, draw s parallel to y; then p, s meet on xy.

All the ring postulates may be verified on this diagram; the pecked lines show that both multiplication and addition are commutative; there is a unity element, namely 1/1; every rational except 0/1 has a multiplicative inverse, namely its reflection in the line 1/1 if positive, in the line $-1/1$ if negative. Hence the rationals form a commutative field.

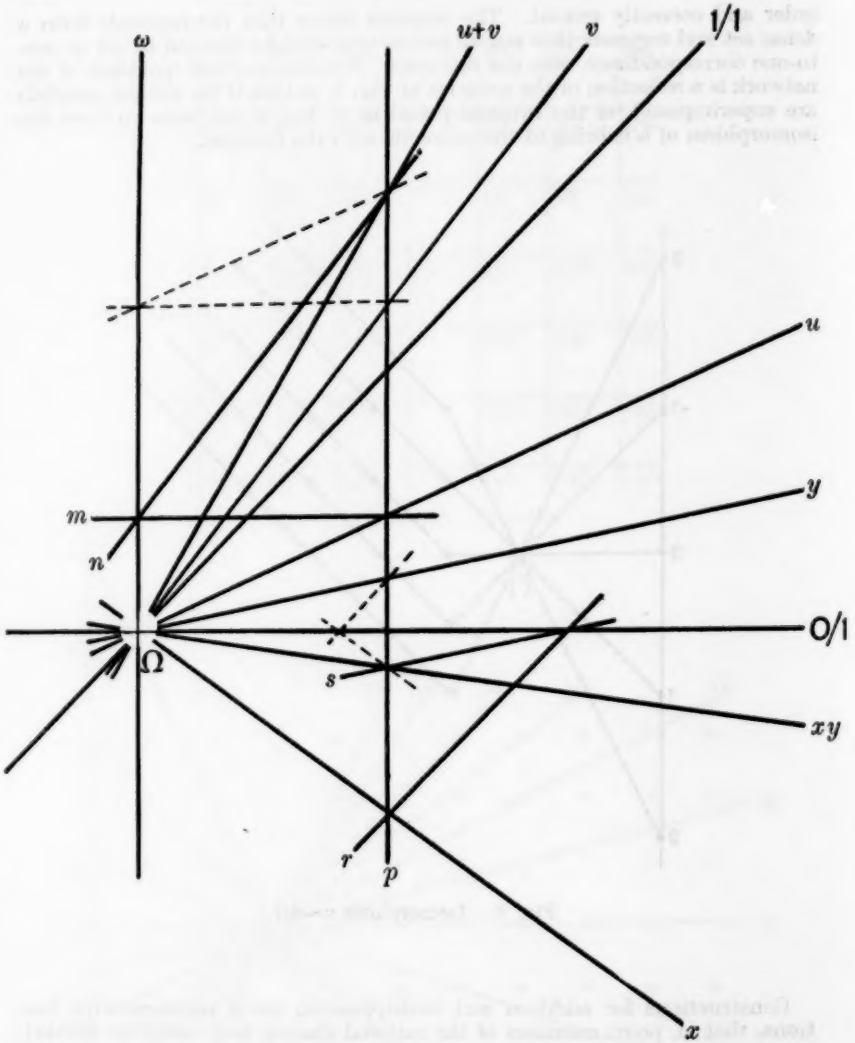

FIG. 7. To add and multiply rationals.

(3) *Quadratic Surds*

Let a, b, c, d now denote rationals, and let f, g, h denote polynomials over the field of rationals, h being the irreducible polynomial x^2-2. Set up the equivalence relation $f=g$ if and only if $f-g$ is divisible by h, writing $f=g \bmod h$. The polynomials fall into classes, called residue classes, each of which may be typified by its unique linear member of the form $a+bx$. We define the sum of the two classes containing $a+bx$, $c+dx$ respectively as the class containing $(a+c)+(b+d)x$; their product as the class containing $(a+bx)(c+dx)$. The

classes thus related form a commutative field. This field contains a subfield isomorphic with the field of the coefficients, in this case the rationals, and *a fortiori* a subring isomorphic with the ring of integers.

Since $x^2 = 2 \bmod h$, x^2 belongs to the class containing 2. Hence we denote the class containing x by the symbol $\sqrt{2}$; that containing $a + bx$ by $a + b\sqrt{2}$; that containing $(a + bx)(c + dx)$, that is, $ac + (ad + bc)x + bdx^2$, by

$$(ac + 2bd) + (ad + bc)\sqrt{2}.$$

The residue classes thus symbolised are termed quadratic surds.

The surd $a + b\sqrt{2}$ may be represented visually by a point with coordinates (a, b), plotted against rectangular axes on which rational scales have been marked. Addition and multiplication of $a + b\sqrt{2}$ and $c + d\sqrt{2}$ are shown in Fig. 8. Here X is (a, b), Y (c, d), I $(1, 0)$. To add, carry out the usual parallelogram construction. To multiply, make triangle LYO similar to triangle XIO; draw LM perpendicular to OY and take $MN = 2\,LM$; then the horizontal through L and the vertical through N meet at

$$P\ (ac + 2bd,\ ad + bc).$$

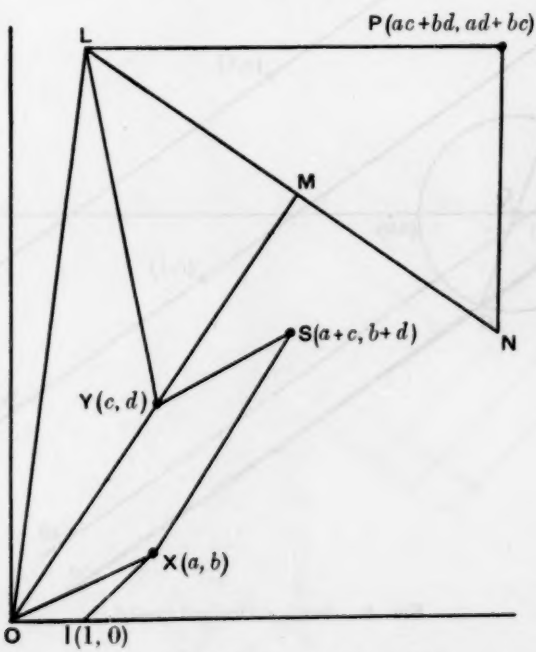

FIG. 8. To add and multiply surds.

The point (0, 1) represents $\sqrt{2}$. It may be verified on the diagram that squaring by the construction just given leads to the point (2, 0), which represents the surd isomorph of the rational integer 2. Again, it may be verified that if $a \neq 0 \neq b$, $a + b\sqrt{2}$ has a multiplicative inverse, namely

$$(a - b\sqrt{2})/(a^2 - 2b^2).$$

We may order the field of surds as follows. $a+b\sqrt{2}=0$ (residue class zero) if and only if $a=b=0$ (rational integer zero); if a, b are of the same sign, we ascribe that sign to $a+b\sqrt{2}$; if $a>0>b$, we ascribe to the surd the sign of a^2-2b^2; if $a<0<b$, we ascribe to the surd the sign opposite to that of a^2-2b^2. Note that $a^2-2b^2\neq 0$ for all rational a, b unless $a=b=0$. We now define ordering relations: $a+b\sqrt{2}\gtrless c+d\sqrt{2}$ according as $(a-c)+(b-d)\sqrt{2}\gtrless 0$.

On the point diagram carry out the construction shown in Fig. 9. The circle centre Q has a diameter of 3 units; the ordinate at (1, 0) cuts the circle in P; PQ cuts the circle again in R; OR is joined. The line OR passes

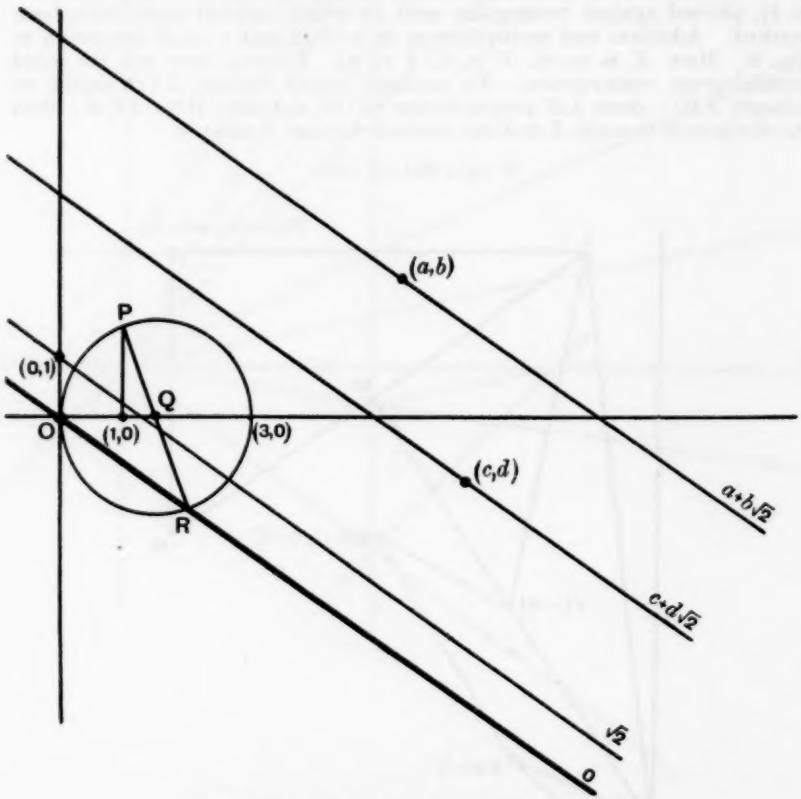

FIG. 9. Surds: Ordered pencil.

through only one point representing a surd, namely the origin (0, 0); all other points (a, b) lie above or below OR according as $a+b\sqrt{2}$ is positive or negative. If $a+b\sqrt{2}>c+d\sqrt{2}>0$ or if $a+b\sqrt{2}<c+d\sqrt{2}<0$, (a, b) is more distant from OR than is (c, d). Now consider parallels to OR through all the points (a, b). Since only one such point lies on each of these parallels, they may be taken to represent the surds. They will cut any transversal, in particular the horizontal axis, in points giving a correctly ordered and spaced distribution of the surds $a+b\sqrt{2}$ in the linear continuum.

Addition and multiplication of the surds, as a pencil of parallel lines, may be carried out exactly as in Fig. 3; but now all the field postulates will be fulfilled. For example, to find the inverse of $a+b\sqrt{2}$ $(a \neq 0 \neq b)$, we may take for m the vertical axis, for n the horizontal axis; draw r through the meets of $a+b\sqrt{2}$ and the horizontal axis, of $1+0\sqrt{2}$ and the vertical axis; where $1+0\sqrt{2}$ meets the horizontal axis, draw s parallel to r; then s meets the vertical axis on the line of the pencil representing $(a-b\sqrt{2})/(a^2-2b^2)$.

If we superimpose directly the integer pencil of Fig. 1 and the rational pencil of Fig. 5 on the surd pencil of Fig. 9, several isomorphisms may be illustrated by showing the conics determined by related pencils. See Fig. 10. Here the ring of integers (n) intersects the subring of rationals $(n/1)$ in points of the hyperbola Σ; the same ring of integers (n) intersects the subring of surds $(n+0\sqrt{2})$ in points of the horizontal axis Σ'; the field of rationals (p/q) intersects the subfield of surds $(p/q+0\sqrt{2})$ in points of the hyperbola Σ''.

The process of forming residue classes from polynomials over the rationals may be carried out with $h=x^2-k$ where k is any non-square positive integer,

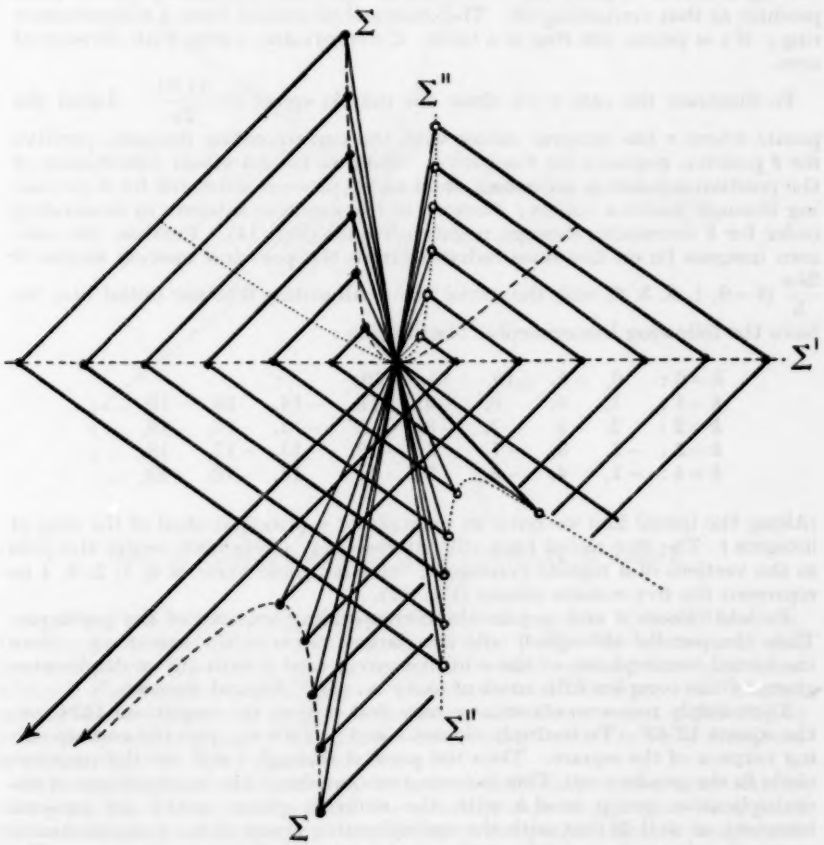

FIG. 10. Isomorphisms $n \sim n/1$; $n \sim n+0\sqrt{2}$; $p/q \sim p/q+0\sqrt{2}$.

giving residue classes of the form $a+b\sqrt{k}$. In Fig. 8 we take $MN=k\,LM$; in Fig. 9 the circle is drawn on a diameter of $k+1$ units. Again, we may form the residue classes for successive k, in each case after the first taking polynomials over the field last formed. We may thus form any quadratic surds and display their ordering, whether as points of a plane, as points distributed in the linear continuum, or as a pencil of parallel lines. Note that if we take $h=x^2+1$, the residue classes are complex numbers, and the point representation of Fig. 8 becomes the " Argand diagram ", but now no ordered display is possible.

(4) *Congruent Integers*

Let a, b, c denote integers, where $c>1$. Set up the equivalence relation $a=b$ if and only if $a-b$ is divisible by c, writing $a=b \bmod c$. The integers fall into classes, called residue classes, c in number; their members leave on division by c the remainders $0, 1, \ldots, c-1$ respectively. We may denote the classes by these c representative integers. The sum of the two classes containing a and b respectively is defined as the class containing $a+b$; their product as that containing ab. The classes thus related form a commutative ring; if c is prime, the ring is a field; if c composite, a ring with divisors of zero.

To illustrate the case $c=5$, draw the double spiral $r=\dfrac{5\,|\,\theta\,|}{2\pi}$. Label the points where r has integral values with the corresponding integers, positive for θ positive, negative for θ negative. Then we have a visual distribution of the positive integers in ascending order as we proceed outwards for θ increasing through positive values; likewise of the negative integers in descending order for θ decreasing through negative values (Fig. 11). Further, the non-zero integers lie on five lines radiating from the pole and making angles of $\dfrac{2k\pi}{5}$ $(k=0, 1, 2, 3, 4)$ with the initial line. Allocating 0 to the initial line, we have the following homomorphic distribution:

$k=0$:	0,	±5,	±10,	±15,	±20,	...;			
$k=1$:	1,	-4,	6,	-9,	11,	-14,	16,	-19,	...;
$k=2$:	2,	-3,	7,	-8,	12,	-13,	17,	-18,	...;
$k=3$:	-2,	3,	-7,	8,	-12,	13,	-17,	18,	...;
$k=4$:	-1,	4,	-6,	9,	-11,	14,	-16,	19,	....

(Along the initial line we have an example of a principal ideal of the ring of integers.) The five radial lines will intersect any circle with centre the pole in the vertices of a regular pentagon; we label these vertices 0, 1, 2, 3, 4 to represent the five residue classes (Fig. 12).

To add classes x and y, join the corresponding vertices of the pentagon. Then the parallel through 0 cuts the (larger) circle in the sum $x+y$. Note the visual isomorphism of the additive group mod 5 with the multiplicative group of the complex fifth roots of unity on the " Argand diagram ".

To multiply non-zero classes, we may first project the trapezium 1234 into the square $12'43'$. To multiply classes x and y, $x\neq0\neq y$, join the corresponding vertices of the square. Then the parallel through 1 will cut the (smaller) circle in the product xy. This last construction shows the isomorphism of the multiplicative group mod 5 with the additive group mod 4 (in rational integers), as well as that with the multiplicative group of the complex fourth roots of unity ±1, $\pm i$ (the units of the ring of Gaussian integers). The correspondences are

Multiplication mod 5	Classes	1	2	4	3
Addition mod 4	Classes	0	1	2	3
Multiplication	Numbers	1	i	-1	$-i$.

Acknowledgments and Notes

The theory summarised at the beginning of each section and at the end of Section (3) will, of course, be found much more generally treated in standard works on modern algebra. The diagrams, with the exception of the first, are original.

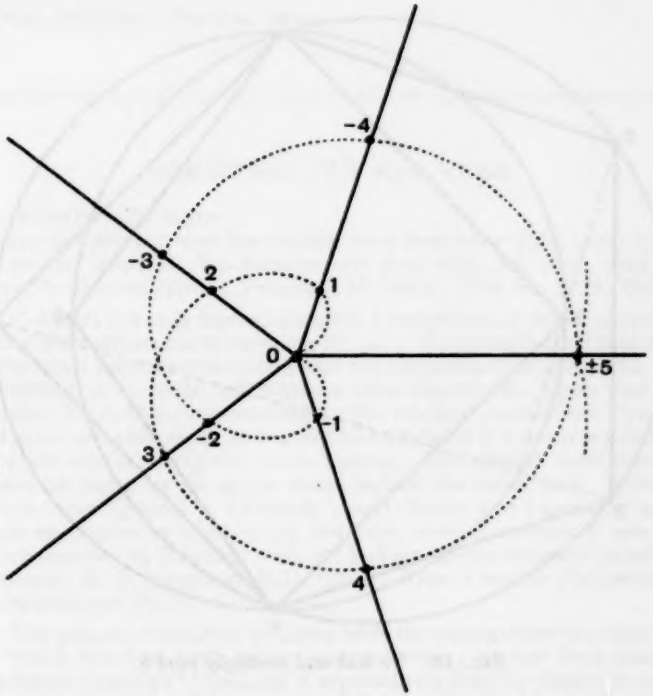

FIG. 11. Homomorphism : Spiral distribution of integers.

(1) Fig. 1 is a slight development of the illustration on p. 10 in N. Jacobson, *Lectures in Abstract Algebra*, Vol. I, New York, 1951. The network of this figure is put to interesting use in connection with continued fractions (after Klein) in H. Davenport, *The Higher Arithmetic*, London, 1952, pp. 111–113, and a comparable diagram is used in connection with Farey series (after Pólya) in G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 3rd edit., Oxford, 1954, pp. 26–29.

The constructions of Fig. 3 may be proved metrically ; they may also be interpreted as affine constructions, the quadrilaterals *mnpq*, *mnrs* being used for addition and multiplication respectively on the pencil of parallels in which the lines 0, 1 are given and ω is the line at infinity. For an explanation of the dual constructions (quadrangles on a range of points) see C. W. O'Hara and

D. Ward, *An Introduction to Projective Geometry*, Oxford, 1937, pp. 152, 153.

(2) In Fig. 4 the ordered pairs are not Cartesian coordinates; in Fig. 5 the rational p/q is represented by a line of gradient p/q. The Cartesian interpretation of Fig. 6 would be: the integer n is represented by $y=x+n-1$, the rational $n/1$ by $y=nx$; the conic determined by the intersections of corresponding rays of the pencils is the line-pair $(x-1)(x-y)=0$.

The constructions of Fig. 7 are exactly the same, from the affine point of view, as those of Fig. 3. For addition we use the quadrilateral consisting of m, n, p and the line at infinity; for multiplication that consisting of r, s, p and the line at infinity; in the pencil the lines 0, 1 and ω are all given.

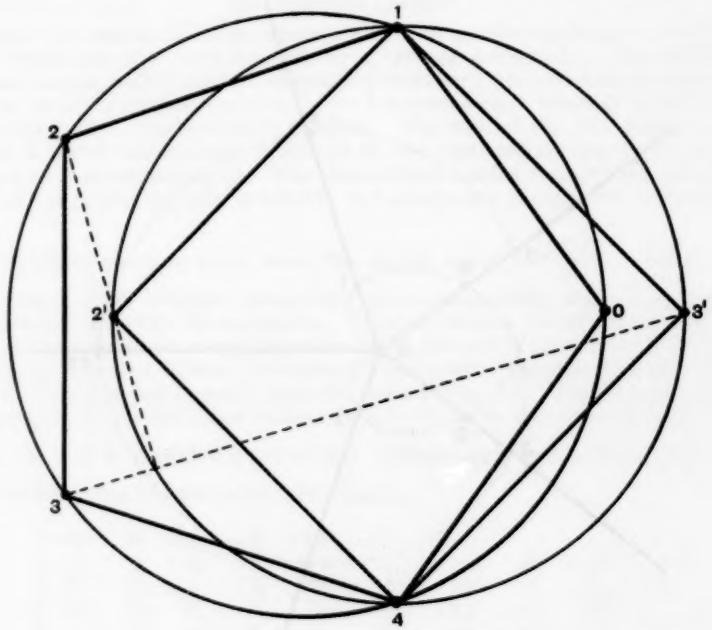

FIG. 12. To add and multiply mod 5.

(3) With regard to Fig. 8 notice that polynomials over a ring are defined as equivalence classes of finite ordered sets of ring elements. See W. V. D. Hodge and D. Pedoe, *Methods of Algebraic Geometry*, Vol. I, Cambridge, 1947, pp. 19, 20. Thus $a+bx$ (a, b rational) is merely a conventional way of writing $(a, 0)+(b, 0)(0, 1)=(a, 0)+(0, b)=(a, b)$. Hence the point representation of Fig. 8.

An alternative representation of the surds in the Cartesian plane is obviously by all lines $y=a+bx$ with a, b rational. (The Cartesian language is convenient, but the representation could be built up from Fig. 4; the pencil of Fig. 5 would then represent the "monomial surds" $0+bx$). The ordering of the surds would be as follows: $y=0$ cuts the line $x=\sqrt{2}$ at the point $(\sqrt{2}, 0)$; all other lines $y=a+bx$ cut $x=\sqrt{2}$ either above $(\sqrt{2}, 0)$ and represent positive surds, or below and represent negative surds. Constructions for addition and multiplication may be devised in this representation, but multiplication is rather complicated.

The line OR in Fig. 9 has Cartesian equation $x+y\sqrt{2}=0$; if this line or any parallel passed through more than one point with rational coordinates, $\sqrt{2}$ would be rational.

In Fig. 10 the Cartesian equations of the various subrings of lines are: for the integers (i) $x-y=n$; for the rationals (ii) $y=nx$ and (iii) $y=px/q$; for the surds (iv) $x+y\sqrt{2}=n$ and (v) $x+y\sqrt{2}=p/q$. Corresponding rays of (i) and (ii) intersect on Σ: $x^2-xy-y=0$; those of (i) and (iv) on Σ': $y=0$; those of (ii) and (iv) and those of (iii) and (v) on Σ'': $x^2+xy\sqrt{2}-y=0$.

(4) The spiral distribution of Fig. 11 and the addition of cyclic points in Fig. 12 can be adapted to any modulus c; the construction for multiplication is clearly not so adaptable.

Oak Mount, Gib Lane, Hoghton, Lancs. T.D.

GLEANINGS FAR AND NEAR

1890. NEGATIVE RABBITS?

"'There are 300 per cent less rabbits here than were eight years ago when I came to the district. We farmers can deal with our own', said Mr. G. Creighton."—*Timaru Herald*, February 12, 1955. [Per Mr. M. A. Bull.]

1891. "About the only factual question I remember in the whole series was one asking the mathematical meaning of 'pi'. Gould explained that it represented the ratio between the radius and the circumference of a circle. Asked by McCullough if he could tell listeners what exactly the figure was he said that roughly it could be represented by the fraction twenty-two over seven. He went on to say that as far as it could be calculated it was three point..... and he then added thirty-five more figures. McCullough took these down on a piece of paper as he spoke them and at the finish said, 'Gould, can you repeat those figures?' 'Certainly', said Gould, and proceeded to do so. I was not so impressed afterwards, however, when I worked it out myself. 'Pi' is represented by 3·142857 with the last six figures repeated to infinity!"
—Commander A. B. Campbell, R.D. (1953), *When I was in Patagonia*. [Per Mr. N. Cannell and Mr. G. A. Garreau.]

1892. The process of coming to terms with the unconscious is a true labour, a work which involves both action and suffering. It has been named the "transcendent function"[1] because it represents a function based on real and "imaginary", or rational and irrational, data, thus bridging the yawning gulf between conscious and unconscious.

Footnote to the above

[1] I discovered only subsequently that the idea of the transcendent function also occurs in the higher mathematics, and is actually the name of the function of real and imaginary numbers.

From C. Jung. *Two Essays on Analytical Psychology, Collected Works*, Vol. 7. [Per Prof. J. L. B. Cooper.]

1893. The borough of Vondervotteimittiss has existed, from its origin, in precisely the same condition which it at present preserves. Of the date of this origin, however, I am grieved that I can only speak with that species of indefinite definiteness which mathematicians are, at times, forced to put up with in certain algebraic formulae. The date I may thus say, in regard to the remoteness of its antiquity, cannot be less than any assignable quantity whatsoever."—E. A. Poe, *The Devil in the Belfry*. [Per Mr. B. J. Barnes.]

NEGATIVE PEDAL OF THE ELLIPSE WITH RESPECT TO A FOCUS

By E. H. Lockwood

1. My attention was first called to this attractive curve by a pupil, M. J. Burleigh ; so, lacking any other name, we called it " Burleigh's Oval ". It is mentioned in Salmon's *Higher Plane Curves* (p. 107), where a cartesian equation is given ; and in Hilton's *Plane Algebraic Curves* (p. 64), where it is described, with a diagram, as the reciprocal of the Limaçon. It also occurs in a very interesting general treatment of negative pedals by A. Ameseder, (*Archiv der Mathematik und Physic*, LXIV, 1879).

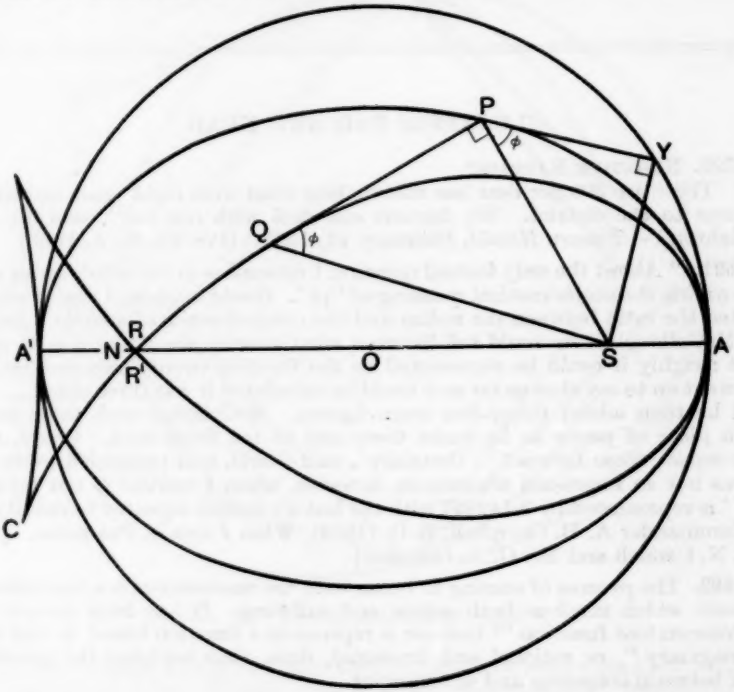

Fig. 1.

Given a circle of radius a, the curve may be drawn with the aid of a set square only. Let S be a point distant ae from the centre of the circle ($e<1$). If Y is any point on the circle and YP is drawn at right angles to SY, YP envelops an ellipse of eccentricity e. If P is on the ellipse, and PQ is drawn at right angles to SP, PQ envelops Burleigh's Oval. If $e \leqslant \frac{1}{2}$ this is an egg-shaped curve on an axis of length $2a$; but if $e > \frac{1}{2}$, it has a node and two cusps, as shown in Fig. 1.

2. *Cartesian Equation.* If AA' is the major axis of the ellipse, A being the vertex nearer to S, the polar equation of the ellipse, with SA as initial line, is $l/r = 1 + e\cos\theta$. The required curve is the envelope of

$$x\cos\theta + y\sin\theta = r,$$

where x and y are rectangular cartesian coordinates with S as origin and SA as axis of x.

Eliminating r, $$x\cos\theta + y\sin\theta = \frac{l}{1+e\cos\theta}.$$

Differentiating w.r. θ, $$-x\sin\theta + y\cos\theta = \frac{le\sin\theta}{(1+e\cos\theta)^2}.$$

Hence $$x = l\,.\,\frac{\cos\theta + e\cos 2\theta}{(1+e\cos\theta)^2} \quad \text{and} \quad y = l\,.\,\frac{\sin\theta + e\sin 2\theta}{(1+e\cos\theta)^2}.$$

These are parametric equations for the curve and the Cartesian equation may be obtained by eliminating θ:

Putting λ for $1/(1+e\cos\theta)$, we have

$$(x^2+y^2)/l^2 = (e^2-1)\lambda^4 + 2\lambda^3 \quad \text{and} \quad ex/l = 2 - 3\lambda + (1-e^2)\lambda^2.$$

Eliminating λ, an equation of the fourth degree is obtained:

$$(x^2+y^2)^2(1-e^2)^3 + 2e^2x^2(x^2+y^2)(1-e^2)^2 + elx(x^2+y^2)(1-e^2)(10+8e^2) + l^2(x^2+y^2)(8e^4+20e^2-1) - 2l(2l-ex)^2 + (1-e^2)(2l-ex)^4 = 0.$$

3. *Curvature, Node and Cusps.* Results are more easily gained from the p, ψ equation, obtained from the polar equation of the ellipse by substituting $r = p$ and $\theta = \psi - 90°$. Thus

$$p = \frac{l}{1+e\sin\psi}, \qquad \text{.....(1)}$$

and hence

$$\frac{dp}{d\psi} = \frac{-le\cos\psi}{(1+e\sin\psi)^2} \quad \text{and} \quad \frac{d^2p}{d\psi^2} = \frac{le(2e+\sin\psi - e\sin^2\psi)}{(1+e\sin\psi)^3}. \qquad \text{.....(2)}$$

The radius of curvature is given by

$$\rho = p + \frac{d^2p}{d\psi^2} = l\,.\,\frac{(1+2e^2)+3e\sin\psi}{(1+e\sin\psi)^3}. \qquad \text{.....(3)}$$

Angle SQP = angle $SPY = \phi$ (as may be seen by considering neighbouring points P, P' on the ellipse and lines PQ, $P'Q$ at right angles to SP, SP' respectively) and

$$\cot\phi = \frac{1}{r}\frac{dr}{d\theta} = \frac{e\sin\theta}{1+e\cos\theta} = \frac{-e\cos\psi}{1+e\sin\psi}. \qquad \text{.....(4)}$$

The node N is given by $\psi = 180° + \phi = 90° + \theta$. Substituting in (4),

$$\cot\psi = \frac{-e\cos\psi}{1+e\sin\psi},$$

whence $$\sin\psi = -1/2e \quad \text{and} \quad p = 2l. \qquad \text{..... (5)}$$

$SN = p\operatorname{cosec}\phi = 4el$, and the radius of curvature at the node is $4l(4e^2-1)$.

The cusps are given by $\rho = 0$, whence

$$\sin\psi = -(1+2e^2)/3e \qquad \text{.....(6)}$$

and $$p = \frac{l}{1-(1+2e^2)/3} = \frac{3l}{2(1-e^2)} = \frac{3a}{2}.$$

The distance R from the focus to either cusp C is given by

$$R^2 = p^2 \operatorname{cosec}^2 \phi = \frac{27a^2}{16(1-e^2)}, \quad \text{using (4).}$$

The distance between the cusps is $2(p\cos\psi - p\cot\phi\sin\psi)$, where p, ψ have the values given by (6). This is equal to

$$2 \cdot \frac{3a}{2}\cos\psi\left(1+\frac{e\sin\psi}{1+e\sin\psi}\right) = \frac{a(4e^2-1)^{\frac{3}{2}}}{2e(1-e^2)^{\frac{1}{2}}}.$$

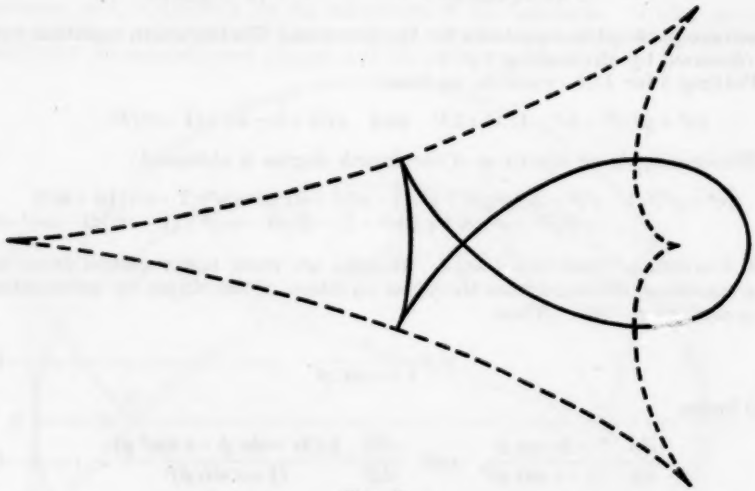

FIG. 2.

4. *Evolute.* If p', ψ' refer to the evolute,

$$\psi' = 90^\circ + \psi \quad \text{and} \quad p' = PQ = \frac{dp}{d\psi} = \frac{-le\cos\psi}{(1+e\sin\psi)^2} = \frac{-le\sin\psi'}{(1-e\cos\psi')^2}.$$

Hence $$\frac{dp'}{d\psi'} = -l[\mu - 3\mu^2 + 2(1-e^2)\mu^3], \quad \text{where } \mu = 1/(1-e\cos\psi').$$

and $$\frac{d^2p'}{d\psi'^2} = le\sin\psi[\mu^2 - 6\mu^3 + 6(1-e^2)\mu^4].$$

The radius of curvature of the evolute p' is given by

$$\rho' = p' + \frac{d^2p'}{d\psi'^2} = le\sin\psi[-6\mu^3 + 6(1-e^2)\mu^4].$$

Cusps of the evolute are given by $\rho' = 0$, whence

$$\psi' = 180^\circ \text{ or } 360^\circ \text{ or } \cos^{-1} e,$$

and

$$\psi = 90^\circ \text{ or } 270^\circ \text{ or } \sin^{-1}(e).$$

The evolute therefore has four cusps, corresponding to points A, A', R, R' of the oval. Of these, R and R' are symmetrically placed, since ψ increases from 90° to 270° as we traverse the curve from A through Q, N, C to A'; and from 270° to 450° as we continue from A' to A.

The radius of curvature has a minimum value at A and maximum values at R, R'. At A' it has a minimum value if $e \leqslant \frac{1}{2}$ and a maximum value (in absolute magnitude) if $e > \frac{1}{2}$.

The length of the evolute is given by $\int p'\,d\psi' + \dfrac{dp'}{d\psi'}$. Measuring from the point corresponding to A, this is equal to

$$l\left\{\frac{1-3e\cos\psi'+2e^2}{(1-e\cos\psi')^2} - \frac{1+2e}{(1+e)^2}\right\}.$$

At $\psi' = 360°$ (for A') this is $-4ae^2/(1-e^2)$, and at $\psi' = \cos^{-1} e$ it is

$$ae^2(3-2e)/(1-e^2).$$

But ds' changes sign at the cusp of the evolute: therefore the whole length of the evolute is

$$2\left\{\frac{2ae^2(3-2e)}{1-e^2} + \frac{4ae^2}{1-e^2}\right\}, \quad \text{or} \quad \frac{12ae^2}{1-e^2}.$$

5. *The special case* $e^2 = \frac{1}{2}$ (illustrated in Figs. 1 and 2) is of some interest. The node is rectangular and coincides with the other focus of the ellipse. The distance between the cusps is equal to the maximum width of the other part of the curve, both these distances being half the major axis of the ellipse. The cusp tangents correspond to points on the ellipse collinear with N. The total length of the curve is $a\sqrt{2}(\frac{1}{2}\pi + 3)$ and the area is $\frac{4}{3}a^2$. The points R, R' coincide with N, the corresponding radius of curvature being $2a$, and the corresponding cusps of the evolute are collinear with S. The length of the evolute is $12a$. The Cartesian equation, referred to N as origin, becomes

$$(2x^2+y^2)^2 - 2\sqrt{2}ax(2x^2-3y^2) + 2a^2(y^2-x^2) = 0.$$

E.H.L.

PSEUDARIA

17. "*Therefore* $0=0$ *and the identity is true.*"

Suppose $2=7$

Then, clearly, $7=2$

$\therefore\ 9=9$

which is true, $\therefore\ 2=7$

A.P.R.

18. *Perimeter of an Ellipse.*

The area of an ellipse (a, b) is πab. The area of an ellipse $(a+h, b+h)$ is $\pi(a+h)(b+h)$. But the difference between the areas is an elliptical annulus whose width at the intersections by the axes is h. The area of this annulus is therefore approximately ph, where p is the perimeter.

$$\therefore\ ph = \pi(a+h)(b+h) - \pi ab = \pi(a+b)h + h^2$$

$$\therefore\ p = \pi(a+b) \text{ when } h \to 0.$$

(Check: if $a=b$, the ellipse becomes a circle, and $C = 2\pi a$.)

A.P.R.

SCHOOL MARKS *

By B. C. Brookes

My main thesis is that men have too long allowed themselves to be bemused by the magic of numbers, though the kinds of spell that numbers have cast over us have changed with the ages. In seeking to understand why it is difficult for us to break the spell, I have been led to suspect that schoolmasters (and perhaps especially schoolmistresses) are largely to blame because of the way in which they use school marks. And a special responsibility in this matter rests upon the mathematical staffs of schools, not because they are more " mark-minded " than their colleagues, but because marks appear to be numbers or measures of some kind and the non-mathematicians must look mainly to the mathematician for guidance on the manipulation and interpretation of numbers and measures. Any lead that may be needed to reform our school marking systems must be given, if it is to be given at all, by the mathematicians. That is my justification for addressing my comments on School Marks to the Mathematical Association.

Now I am well aware that an enquiry into marking, if taken really seriously, could lead us into very deep waters, into the murky depths of psychology, of epistemology, of aesthetics, perhaps even of metaphysics. But you may be relieved to hear that I propose only to splash about in the shallow waters of statistics—and then not too seriously.

But, on the principle that it is easier to see the beam in our own eyes if we first look at the moat in the eyes of others, I am going to begin by reminding you of some mathematical history. We sometimes forget that the numerals, the so-called " Arabic " numerals, we use for school marks (and for other purposes) are very new toys. We in Western Europe have played with them for only a few hundred years and we are still fascinated by their beady objectivity. To help you to understand what I mean let us first look, very briefly, at an earlier system of numerals and at some earlier number mysticisms that developed from them.

The earliest known number systems were purely representative—one stroke for one thing, two strokes for two things, and so on—they were simple but cumbrous to use. I doubt whether Babylonian schoolmasters indulged in orgies of form orders.

But the systems which superseded them were much more sophisticated and interesting, no longer simple one-to-one representations but systems of conventionalised symbols. They were used by the early civilisations of what we now call the Near East, for example, the Jews, the Phoenicians and the Greeks. Have you ever considered how there comes to be such a thing as an alphabetical order—A, B, C ...? Order for what?—apart from arranging telephone directories, dictionaries and, of course, form lists? Why should you when you ask a child to " say his letters " insist that he says them in that one permitted order which begins with A and ends with **Z**? Do you realise, when you do that, that you are really teaching him his " numbers "? —his Greek numbers? All the civilisations I mentioned used the letters of their alphabet as symbols for their numerals, and of these the later Greek system is the best-known.

As you can see, the system is straightforward. The Greek alphabet had only 24 letters, but, by borrowing three letters from the Phoenician alphabet, the Greeks were able to represent all the numbers from 1 to 999 with their 27 letters. By adding suffixes to the symbols it was possible to represent

* A lecture delivered at the Annual General Meeting of the Mathematical Association, January, 1956.

numbers greater than 1,000, though large numbers were used only rarely. When I consider this numeral system it never surprises *me* that Greek mathematicians showed more aptitude for geometry than for arithmetic, though I wonder the more at the genius of Archimedes and of Diophantos. It also seems to me to demonstrate how important it is to choose your symbols rightly when beginning a new abstract study. But the point I am at present interested in is not the fact that the system was awkward to use, but that the Greeks used the letters of their *alphabet* as the symbols of their numbers, putting only a bar over the letter to indicate that it was to be read as a number. This fact led to a curious result. Whenever a Greek wrote his *name* he also wrote his number. No question then of giving your name *and* number —your name (in a sense) was your number. It is therefore perhaps not surprising to find that among the Greeks and the other civilisations that used similar numeral systems there developed a mystic cult about numbers.

When I was a boy of 10 or 11 years of age, having just been initiated into the mysteries of algebra, which at that stage seemed to consist solely of exercises of the kind : " Find the value of $ab^2c + abc^2 - a^2bc$ when $a = 1$, $b = 2$, $c = 3$ ", I and my fellow schoolboys invented a game. We began by writing down the names of our friends and of our teachers, together with a list of epithets, some flattering, some derogatory. The problem then was to find a number code, such as $a = 3$, $b = 4$, $c = 2$, and so on—integral values—which, on substitution for the letters in the names and epithets, would produce totals which equated the names of our friends with the flattering epithets and, with the same code, the names of our teachers with the derogatory epithets.

We did not know then that this game of ours was in fact a revival of the very ancient pseudo-science called Gematria. As far as I know the principles of Gematria were never explicitly stated. They were, I suppose, so obvious to anyone who used his alphabet for numbers that they were never questioned. The basis of Gematria was the principle of the equivalence of names or words which had the same numbers. For example, Plato's number is :

$$\begin{aligned}\pi + \lambda + \alpha + \tau + \omega + \nu &= 80 + 30 + 1 + 300 + 800 + 50 \\ &= 1261.\end{aligned}$$

If we now were to find the numbers corresponding to the words describing all human qualities we would determine Plato's character by noting which of them had the same, or nearly the same, number as Plato—and those would represent Plato's characteristics. You may perhaps appreciate that some—shall we say?—adjustment might be necessary to endow your friends and your enemies with the qualities you think appropriate, but this fudging went on seriously and happily right up to the sixteenth century—to the time of the Reformation. In the religious feuds of that period there were earnest theologians on both sides who devoted themselves to proving that the name-number of the opposing protagonists was the most dreadful of all—the dreaded number 666, the so-called number of the " Beast ". Knowledge of this fact would, of course, have added both zest and an element of determinacy to our schoolboy game.

Gematria is just one example of the mysticisms about numbers that have fooled intelligent people in the past. Its study lasted for nearly 2000 years and survived until only three or four hundred years ago. It arose out of an accident, a convention to use the letters of the alphabet to represent numbers, and it was killed only by the gradual adoption of the more convenient and non-alphabetical symbols of the Hindus. We can now dismiss the mysticism of Gematria with an indulgent smile. Or can we?

Another ancient source of number mysticism was Pythagoras and his school. Pythagoras had a dictum " Number rules the universe ", echoes of which, in the form " God is a mathematician ", have been heard in more

recent times in Cambridge. To Pythagoras the number 10 was particularly significant ; for example, if you count up to 4 you already have 10 :

$$1+2+3+4=10.$$

Please don't ask me *why* this fact was considered so remarkable ; it was just one of the more intelligible properties of the number 10 that made 10 " perfect " to the Pythagoreans. This kind of " perfection " is not, of course, of the kind we meet in the theory of numbers, where a number is said to be " perfect " if it is equal to the sum of its factors, *e.g.* $28=1+2+4+7+14$. That, too, is of Greek origin, but the word " perfect " in the context of the theory of numbers can be exactly and technically defined, and its implications can be, and still are being, analytically explored. The Pythagorean belief in the " perfection " of the number 10 was of a different kind, emotive and mystical. It led to the formulation of a cosmological theory. The heavens, too, are perfect, and they must therefore contain 10 moving heavenly bodies. Unhappily, only 9 were then visible—the sun, the earth, the moon, the five planets Mercury, Venus, Mars, Jupiter, and Saturn, and lastly the sphere in which the fixed stars were embedded. So, to account for the apparent imperfection of the heavens, Pythagoras postulated the existence of a " counter-earth ", a body which was always invisible because it was always obscured by the " central fire ", the fixed centre about which the 10 heavenly bodies moved in that one perfect curve, the circle. The number mysticism of Pythagoras attracted Plato, and it was largely because of Plato's influence that elements of the Pythagorean cosmology were accepted until comparatively modern times. In fact, it was Kepler, a devout Pythagorean, who in 1609 discovered that the orbits of the planets were not circles but ellipses. And in the same year Galileo saw through his telescope such imperfections as the craters on the moon, satellites round Jupiter and dark spots on the sun. The " perfection " theory of cosmology was shattered. The remote planets Uranus, Neptune and Pluto were found later. No one, I think, to-day would agree that the number of planets *must* for *a priori* reasons, be 7, 8, 9, 10, or any other number.

I have reminded you of two of the better-known examples from the past which show how intelligent men have been hypnotised by numbers. The examples I have chosen were intended to be *obvious* examples of error, and I assume that no one present believes that his name *is* his number or that 10, or any other number, is perfect in the Pythagorean sense I have described—though I must confess that I have some affection for e, π, and one or two other numbers. But, on the whole, I think, we can contemplate these regrettable errors of our ancestors with equanimity, conscious of and confident in our superior objectivity. Or can we? Let us turn to the present and look at ourselves.

Even to-day there remain obvious relics of mystical ideas about numbers in superstitions about lucky and unlucky numbers. But these are trivial matters which need not detain us. There are other number mysticisms, much more prevalent and much more insidious, to which I want to draw your attention.

But before I come to school marks, I must say a few words about measurement, about statistics and about the difficulties of marking. Our present attitude to measurement was largely determined for us by the nineteenth-century physicists and chemists who so successfully established the mechanistic theories of matter. Their success was based on their ingenuity in devising the experiments in physical measurements which led in rapid succession to the formulations of the fundamental laws of science which seemed capable, fifty years ago, of accounting in scientific terms for all known physical phenomena. These scientists seemed to know intuitively what phenomena were

amenable to measurement and what were not. Their attitude was epitomised by Lord Kelvin, who said that we understand something only in so far as we can measure it; and this attitude has now permeated almost every field of study—and notably psychology—so that we now try to measure *everything*.

I do not quarrel with this attitude so long as it is clear what we mean by measuring. But it seems to me that we are too ready to assume that simply by assigning a number to something, however arbitrarily, we have a *measure* and that we therefore understand. And even if we have doubts about the validity of our measuring procedures or about our understanding of the results of our measuring we don't stop to think about what we are trying to do, but immediately invoke the aid of statistical techniques to pull out of the hat for us something that we have not put in; and that seems to me like believing in magic. This attitude can, I think, even be detected in the terminology used, for example, by educational psychologists. When they set a test paper and mark it, they usually call their marks the "raw scores", with the implication, I think, that by then applying their statistical normalising and standardising techniques they are able to convert the raw scores into something more refined, more real and meaningful, rather as the alchemists claimed to turn base metals into gold.

I have often observed this reluctance to examine the validity of numerical data combined with haste to apply elaborate statistical techniques and readiness to draw conclusions. Here is one illustration—a story of the war. The flying training of air gunners included air exercises in which they fired a fixed number of rounds at a target carried by a second aircraft. After the exercise the target was dropped on a special site and the number of bullet holes in it was counted. This number, expressed as a percentage of the rounds fired by the gunner, was the gunner's score for that exercise. At the end of the training course each gunner would have several such scores and the average of his scores was computed. The average of the averages of all members of the course was then computed and this number was sent to various headquarters. Here it was recorded on the inevitable chart so that it could be compared visually with similar averages for other courses of the same training establishment and with similar averages received from other training establishments. The recorded scores plotted on the charts showed that the averages of one school were persistently lower by about 40% than the averages of the others. Various theories were propounded to explain the phenomenon and appropriate action was taken:

1. That the technical *instructors* at the school were incompetent. So the best instructors available elsewhere were sent to replace them. Result: no change.
2. That the *administrative* staff was incompetent. So they were all posted and proved administrators were sent to replace them. Result: no change.
3. That the *equipment* was in some indiscernible way at fault. So new aircraft, guns, etc., were sent as replacements. Result: no change.
4. That the *exercises* were improperly carried out. So careful air-borne inspection of them was made. Result: no fault, no change.
5. That the *station personnel* were not sufficiently "on their toes". So inspection teams of very stern and senior officers descended from the clouds at random intervals and without notice. Result: no change.
6. That the station's personnel were deficient in *morale* (perhaps as a result of all these critical inspections). So the whole station was tidied up and repainted. Result: no change.

Eventually, after every other possible cause of the discrepancy had been considered without result, the actual counting of the scores was checked.—

Result : success! The cause of the trouble was that the target-dropping site was inaccessibly located about 40 miles from the station. No technical officer ever went there to see what happened. When the targets were dropped, the identifying numbers were dutifully recorded but the bullet holes were not counted. At the end of the day the corporal in charge telephoned the scores back to the base station. He correctly read out the target identifications and, as he did so, he thought of a number which would do as a score. At the station the number the corporal thought of was recorded, in good faith, on the appropriate official form. Once entered there it thereby became the unquestionable official data on which any relevant action was based. Unfortunately for the corporal, the standard of air gunnery had improved since he had first begun to think of a number ; fortunately for the air-gunners, the bogus scores seemed in no way to affect their keenness or their training.

I tell that story to illustrate what I meant when I spoke of " reluctance to examine the validity of the data combined with haste to apply statistical techniques and readiness to draw conclusions ". I could tell more stories of the same kind and of more recent occurrence. They all indicate an attitude which is more prevalent than it ought to be ; and it arises, I believe, partly from our teaching of elementary mathematics and partly from our use of school marks. But, in order to explain *why*, I must first try to make two distinctions.

You must often have heard the phrase : " That is *mere* statistics ", a phrase which puzzled me for a time. But now I think I know what " mere statistics " is or are. Roughly, one can say that statistics are *valid* numerical data, data that are what they purport to be, *e.g.* the true scores in the story I have just told you ; whereas *mere* statistics are *bogus* numerical data, data that are *not* what they purport to be, *e.g.* the numbers the corporal thought of. And, by extension, I define a *statistician* as one who manipulates and interprets *statistics*, and a *mere* statistician as one who manipulates and interprets *mere* statistics. This distinction is important. It concerns any one of you who averages or orders the numerical data we call marks. And two questions arise :

1. Are school marks statistics or are they *mere* statistics?
2. Are those of you who use them statisticians or are you *mere* statisticians?

The second distinction I have to make is this. The teacher of any school subject has two jobs to do simultaneously ; he has to teach the *technique* of his subject and, at the same time, he has to foster in his pupils what I will call an appreciation of his subject. Of these two jobs the techniques are by far the easier for the teacher to teach and for the learner to learn. The techniques are what might be called the " tricks of the trade ". They can be listed as items in a syllabus and they can be ticked off one by one as the teacher teaches them and the learner learns them. The techniques of any subject are described in its textbooks.

But what I have called the " appreciation " of a subject is much more difficult and more subtle to grasp than its techniques. Admittedly, before you can appreciate the subject in the sense I am trying to describe, you must first know some of its techniques. My term " appreciation " includes, for example, a sense of judgment about the subject ; one who has the appreciation I speak of will be able to judge not only whether a technique is correctly executed, but also whether the material to which the technique is applied is appropriate for its application.

Appreciation thus includes the ability to criticise the application of techniques. It is needed for understanding the limitations of the subject and its relation to other fields of knowledge and to other human activities and experiences.

These two facets, technique and appreciation, can be found in any school subject, but they are more clearly distinguishable in some than in others. The teacher of English, for example, is usually well aware of his two-fold job. On the one hand he has to teach his pupils how to write English sentences, and this he does by telling them the rules of English grammar, the laws of English syntax. All these are matters which can easily be described in words, copied down in note-books, exemplified and illustrated in textbooks. They can be learnt by rote, and can be applied by the pupils to their own writing in and out of school, uncertainly perhaps at first, but at least with a degree of success that can be measured and expressed in marks. All this belongs to the *technique* of English.

At the same time the teacher of English has to try to instil in his pupils an appreciation of English; and this is a much more difficult job to do. It cannot be reduced to rules of thumb, it cannot be explained in simple words, it cannot be taught by dictating notes. This appreciation is acquired, if it is acquired at all, only slowly; its growth cannot be measured by any simple system of setting test-papers and marking them. The teaching of appreciation requires methods different from those used in the teaching of techniques. Appreciation cannot in fact be taught directly; it can only be fostered and encouraged. It needs in the teacher great skill, great patience and sympathetic understanding of his pupils.

In English I think you will agree that the difference between technique and appreciation is clearly discernible. The difference is recognised by teachers of English, even if, as I think they would be the first to admit, they have not in general solved the problem of *teaching* appreciation. There can, in fact, be no *formal* teaching of appreciation; any attempted formalisation produces only another technique. Each teacher can only seek his own solution to the problem as he sees it and as it confronts him in his particular school and classroom.

In the teaching of science the *same* problem arises, but teachers of science usually ignore it. Science, too, consists of techniques and appreciation; and the techniques are usually crammed into note-books and drilled in the laboratories. Teachers of science hope, if they bother at all, that somehow, by stuffing their students with techniques they will gradually acquire an appreciation of science, that is, they will come to realise what Science and the Scientific Method are, what they can and what they cannot do. They hope that by showing their pupils the atoms one by one they will eventually see the world of science as a whole. Some pupils do, of course, but the controversies which periodically rage in the correspondence columns of *The Times* and elsewhere about deficiencies in the education of scientists and engineers arise fundamentally, I think, from the charge I am making here. The techniques of science are taught, and on the whole they are *well* taught at all levels, but the appreciation of science is left to look after itself. The result is that our schools—and universities—produce *technicians* rather than *scientists*, and *mechanics* rather than *engineers*.

In the teaching of mathematics the situation seems to me to be even worse. Mathematics is, I suppose, the most ancient academic subject, and, in the hands of generations of earnest and well-meaning teachers, it has now become a list of techniques and nothing more. The appreciation of mathematics as distinct from its techniques is hardly discernible at all. To the student, mathematics must look like one of the infinite series he has to learn about (a divergent one at that), an unending list of techniques which stretches on for ever and ever, an endless road running into an unknown desert, a road marked only by the milestones that represent successive examination requirements. What *is* Mathematics? What *can* it do? What can it *not* do? These are questions that now only belong to highly sophisticated forms of philo-

sophy. The student who asks them can now only be told to wait and see and to be persuaded to learn still more techniques. Most of us nowadays are content to take from the list of mathematical techniques those that are of practical importance to us and to leave the rest of them to those who are content to go plodding on without asking awkward questions. Except for the very few, mathematics has become reduced to a purely technical subject.

I have tried to distinguish between two facets of teaching—what I have called Technique and Appreciation. Ideally, in all school subjects, the child's progress in techniques and in appreciation should go hand-in-hand. But, too often, I suggest, technique far outruns appreciation, and that thoughtless marking can increase the unbalance. In the search for speedily obtained and reliable marks, we are driven to test only techniques and to ignore the appreciative aspects of our subject, to use the one-word or one-number answer as the material to be marked rather than the essay or the selection of method. It is perhaps beginning to be realised, as many teachers have long maintained, that the marking of technique alone may give results which are more or less repeatable but yet which are false measures of ability. I was interested to read that the Director of the American College Entrance Examination Board has recently confessed in an annual report that his Board " is obliged to continue to consider the possibility of testing ability to write English by having the candidate write English ".

I am not intending to discuss administrative methods of selecting children for secondary education or of running external examinations—that is another subject. Though the demands of administrators affect the use of marks in the classroom, I want to keep our attention firmly on *teaching*—on the use of marks in the school. So I am not discussing systems of scaling marks, nor have I any elaborate methods of analytical marking to suggest. The marks I am talking about are those your pupils see on their exercises, tests, homeworks and reports ; not the marks they never know, but the marks you give as teachers rather than as administrators.

Let us see what kinds of results we get when we mark papers of different kinds. Two years ago I was able to organise a simple experiment. Some teachers of English were asked to mark some English essays written by ten and eleven-year-old children. Before the marking began, the teachers discussed the problem for two hours and eventually agreed on the marking method to be adopted. They then divided themselves into two panels and independently marked forty essays, each panel working together as a committee. When the two sets of marks for the same set of essays were compared the correlation coefficient was found to be 0·72. The English teachers were disappointed ; after their long discussion and their adoption of an agreed marking procedure they had hoped to achieve a much closer agreement.

The same batch of essays was then handed to twelve teachers of mathematics, who, without warning and without discussion, were asked to mark the essays as quickly as possible and within an hour. The mark sheets returned by the mathematicians were divided at random into two piles to represent two panels, and the average marks of each panel were noted. The coefficient of correlation between the two mathematicians' panels was 0·933 which, even allowing for the fact that only forty essays were used, could be shown to indicate a significantly better agreement than the English teachers were able to achieve with a much greater effort. At this result the English teachers were even more dejected ; they felt that they had failed and that the mathematicians had succeeded. Why? Because they took as their criterion of success the degree of consistency between the two panels. So we looked at this more closely ; how had the higher consistency been achieved? We found that the English teachers had seriously attempted to assess such

qualities as " imagination ", whereas the mathematicians, though later admitting that imagination was a quality to be looked for in an essay, had decided, under the pressure of the marking conditions imposed on them, to ignore it. The mathematicians had in fact achieved their high correlation by marking only the more obvious mistakes in punctuation, spelling and grammar. They had marked what was easily marked and had ignored everything else—under the conditions imposed they had no alternative. But, to sum up, I think you can see that consistency had been achieved by the mathematicians only by dodging the problem the English teachers had tried to solve, by marking technique and ignoring appreciation. And the point I intended this to illustrate was that, though for good marking, *i.e.* for producing statistics rather than *mere* statistics, consistency and repeatability are necessary conditions, they are not by themselves sufficient.

But now let us turn to the marking of *mathematical* scripts. I have no doubt, though I have no data at hand to demonstrate it, that the marking of the forty mathematics scripts by two panels of mathematics teachers, who first spent two hours discussing their marking system, would show a much closer correlation than the 0·72 which the English teachers achieved, *i.e.* the marking of mathematical scripts can be highly consistent. But is this enough? I have some evidence to suggest that there are other aspects of marking to be considered. On the next slide there are some mathematics marks displayed. I have promised not to divulge the source of these marks, but I can tell you that they are in fact marks obtained by a batch of examination candidates in two separate papers on successive days. The two papers were each carefully designed to cover the whole examination syllabus and were intended to be of equal difficulty. There are therefore two marks for each candidate, one for Maths I and one for Maths II. These I have plotted in the usual way as points (x, y) on graph paper. I can assure you that the marking of each paper was carefully planned by the examiners, meticulously performed, and then scrutinised by assessors, so that if either set were remarked according to the same scheme we could reasonably expect closely similar marks. But it is interesting to see what happened when the highly reliable marks of Maths I are plotted against the highly reliable marks of Maths II. Ideally we should expect to see a straight line, but what do we find in fact?—a scatter of points with a correlation coefficient of 0·52.

What do we do about it? Which of these two highly reliable but very different sets of marks is the true one? We don't know, and so we compromise by adding them together.

There is much more evidence of this kind available if you look for it. There is no wholly reliable means of testing ability in any school subject. When we set a test paper of any kind it is a sample of all the test papers we could have devised for the same purpose and circumstances. When we devise a marking scheme, the one we select is a sample of all the similar marking schemes we could have selected. When a schoolboy writes an essay or does some mathematics his script is a sample of all his possible performances on the same paper under the same conditions. When we mark the scripts the mark we give is a sample of all the marks we could equally well give to the same script. The mark we finally enter in our mark-books or reports is a sample of a sample of a sample of a sample, of samples drawn at random from populations whose characteristics are unknown. The wonder is *not* that the mark is unreliable, but that it is of any value or use at all.

With these considerations in mind it is difficult to justify the claim that is implicit in the giving of a *percentage* mark, that is to have measured something, we don't really know what, correctly to two significant figures. Or that in a class of 30 schoolboys we can discriminate between 101 degrees of attainment or ability in any specific subject. Equally, it is difficult to justify

the adding of marks in French to marks in Art or Mathematics, as though adding them together was a meaningful operation.

Yet when I was a schoolmaster a few years ago, we took marks from all kinds of sources, added them, averaged them, arranged them in order and gave the boy who got the highest score the form prize or a scholarship. A boy who got 73% was better than a boy who got 72%—in fact about 1·4% better—and parents and headmasters were distressed if a boy's geography mark slumped from 50% in one term to 49% in the next. Indeed, the only point at issue was the cause of this decline. Was it caused by a waning of intellectual ability or by a weakening of moral fibre? The mark itself must never be questioned once it appears on the official form. Is it surprising that boys grow up reluctant to question the validity of any numerical data that is presented to them on the official form?

I said earlier that, apart from questions of marks, our teaching of mathematics also helped to encourage the unquestioned acceptance of numerical data. So many of the elementary exercises we set in order to teach techniques, especially in mathematics and especially in arithmetic, demand the unquestioning acceptance of the data: "Divide this by that"; "Find the cost of so and so at so much per lb."; "A block of wood of mass so much rests on a plane inclined at so many degrees to the horizontal", and so on; thousands of thousands of them in the course of a boy's school career. The correct data are those printed in the book or written on the board; they are official and everything else is wrong.

Now I agree that my criticisms can be pressed too far; after all, I fully admit, the elementary techniques must be taught. One might suggest that the situation could be met by an occasional discussion with the class about the need to make sure that the basic data are sound before one spends time and energy on them in futile operations or drawing firm conclusions from them. But the teacher who attempts to do this has some difficulties to contend with. Let us try to look at this problem from the schoolboy's point of view.

Over his hasty breakfast the schoolboy may be able to glance at the morning paper. If he does so he will, I think, skip the mere statistics about inflation, imports and the cost-of-living and turn to the less predictable statistics he will find in the sports pages.

Cricket: There always seems to be a test-match going on somewhere nowadays, and our schoolboy may read that one of his batsman heroes scored 99 runs and was then bowled. What is his first reaction? Not "Well played, sir!"—but "Bad luck—just missed his century by one run!"

His cricketing hero, although he succeeded in scoring as many as 99 runs, failed by 1 run to reach that perfection of the batsman—the century. Shades of Pythagoras!

Athletics: To the bewilderment of our European friends who use the metric system, the Anglo-Saxon world of sport has in recent years been striving to reach a new perfection in middle-distance running. The question was: Who will first achieve the feat of running the very arbitrary distance we call one mile in the very arbitrary time we call 4 minutes? Happily the question has now been answered. Floreat Roger Bannister! Floreat Pythagoras!

The sports pages are full of this kind of number nonsense, of meaningless totals and pointless averages, of Pythagorean delight in the perfection of round numbers. And I confess I read them as avidly and with as much excitement as anybody. But even here the schoolboy is taught never to question the official data; a run is a run if the umpire says it is; a goal is a goal if the referee says so; and their decisions must never be questioned. I suppose I must have been one of the very few who saw in the recent cricket dispute with Pakistan about umpires the signs of a new hope for mathematics.

Do these things matter? They do and they don't. They don't if you see them for what they are; they are then harmless and amusing. They do matter, however, insofar as they help to create a mystique about numbers, to make people itch to tot up totals for no other reason than that the numbers are there, to apply elaborate statistical techniques to numbers that are not worth a second thought, and then to draw from them conclusions that may be logically valid but are wholly erroneous because they are based on false premises.

Let us return to our schoolboy. Had his way to school taken him down the Strand some twenty or thirty years ago he might have passed by a battered be-medalled old soldier sitting on the pavement and begging for alms. The passers-by were amused by a notice the old soldier had propped up on the pavement and the steady clink of coppers indicated that he had effectively aroused their interest and sympathy. What did the notice say? Here it is on the next slide.

PLEASE HELP

WARS . . .	3
WIVES . . .	2
WOUNDS . . .	6
MEDALS . . .	8
CHILDREN . .	10
TOTAL . . .	31

As I was preparing this slide a friend, a historian, came into my room and looked at my handiwork. He, too, was amused, so I asked him if he could tell me what amused him. "Yes," he said, "isn't it silly! The order surely ought to be: 'Wars, Wounds, Medals, Wives, Children'!" And so our schoolboy reaches school.

The earnest teacher of mathematics might well use the story of this slide to illustrate one of the homilies I have suggested he might give on the misuse of arithmetical operations. But again there are dangers. A suitable time for such a homily might be one of those interminable mornings I can remember on the last day of term. Textbooks have been checked and put away, exercise books have been taken home, there is no point in beginning any new technique and no point in testing any old one, and everyone is waiting—not too patiently—for the bell that will bring release and the joys of a holiday. There is the risk that just as you have made your point the headmaster, secretary, bursar, whoever it may be in your school, will come in with an armful of official documents, for each boy to take home to his parents. The kind of thing that I have in mind looks like the purely imaginary specimen I have on the next slide.

Perhaps you see the kind of danger I mean! Another danger that the reformer of school marking systems has to face is that by *direct* attack on marks, especially attack by the mathematicians, grave suspicions in other members of the staff are aroused. Perhaps they feel that the would-be reformer is, in some secret way, working towards the advantage of his own subject. And so the probability of reforming the system is likely to be increased if, by exploiting any weaknesses of the marking system, the mathe-

matician can use it obviously to serve his mathematical interests, and especially if at the same time he can find some outrageous defence of it. I will briefly describe one of the ways in which this was done—shortly before I left school-teaching.

You may have noticed that, whereas in mathematics it is not unknown for examinees to get 0% or 100% on a paper, it seems, as though by some natural law, impossible for examinees in any *arts* subject. Examination marks, and even more certainly form marks, in History or English or Latin usually range from a little below 35% to a little above 65%. The mathematician can use these phenomena to ensure, for example, that a form order ranging over the usual subjects conforms with the mathematical order and so produce for him forms or sets of maximum homogeneity in mathematical ability. A little statistical theory will help to explain this system.

If x and y are normally distributed variates with standard deviation σ_x and σ_y, respectively, the variate $(x+y)$ is also normally distributed with standard deviation σ_{x+y} given by

$$\sigma_{x+y}^2 = \sigma_x^2 + 2r\sigma_x\sigma_y + \sigma_y^2 \quad \text{.....(1)}$$

where $r = \text{corr}\,(x, y)$.

The variates x and $(x+y)$ are also obviously correlated and, if

$$r' = \text{corr}\,(x, x+y),$$

it can be shown that

$$r' = \frac{1+kr}{(1+2kr+k^2)^{\frac{1}{2}}} \quad \text{.....(2)}$$

$$\text{where} \quad k = \sigma_y/\sigma_x.$$

If $r = 1$, then $r' = 1$, as we should expect. But, even if $r = 0$, we find a positive correlation given by

$$r' = (1+k^2)^{-\frac{1}{2}}. \quad \text{.....(3)}$$

Now let us denote a set of mathematics marks by the variate x. If the marks are expressed as percentages and can be taken to be approximately normally distributed, ranging from 5 to 95, we can take $\sigma_x = 15$. In my school the marks in mathematics were, for some obscure traditional reason, given double weight. And so $\sigma_x = 2 \times 15 = 30$.

Let us similarly denote a set of history marks by the variate y. In this case we will assume that again the marks are normally distributed, but over the more restricted range 35 to 65. We can then take $\sigma_y = 5$ marks.

Let us now assume that 9 sets of marks in non-mathematical subjects are available from the same set of victims, that all these sets are independent and uncorrelated, and that their distributions are similar to that of the history marks. Let the variate z denote the sum of the marks in these nine subjects.

Then $\sigma_z = \sigma_y\sqrt{9}$, by extension of (1) with $r = 0$,
$= 15$.

Hence, from (2), $k = 15/30 = \frac{1}{2}$, and so, from (3),

$$\text{corr}\,(x, x+z) = (1+\tfrac{1}{4})^{-\frac{1}{2}} = 0{\cdot}89,$$

i.e. the expected coefficient of correlation between the mathematics marks and the total or " form " marks would be 0·89—very high.

On the other hand, the expected coefficient of correlation between the history marks and the form marks would be given by :

$$\text{corr}\,(y, x+z) = \text{corr}\,(y, y+\overline{x+z-y}),$$

so that in this case,

$$k^2 = \frac{30^2 + 8 \cdot 5^2}{5^2} = 44,$$

and the coefficient of correlation between the history marks and the form marks is $(1+44)^{-\frac{1}{2}} = 0{\cdot}14$, a low figure which becomes significant only for very large sets of victims.

Even if the mathematics marks are not doubly weighted the correlation coefficients are still markedly different. We get :

$$\text{corr (maths, form)} = 0{\cdot}71$$

$$\text{and corr (history, form)} = 0{\cdot}23$$

We have, of course, assumed that the sets of marks are uncorrelated, though in practice they always are correlated to some extent. Correlation among the non-mathematical subjects will reduce the value of corr (maths, form), but correlation between the mathematics and non-mathematics marks will increase that value. The effects of correlations usually found between sets of school marks broadly cancel each other.

If teachers of mathematics apply this method to organise forms and sets to conform with their requirements, it is apt to be noticed and should give rise to suspicion in the common-room. The mathematician's next step, if any further step is needed, is to claim that the order obtained is perfectly natural since, by taking form-order as the best available measure of general intelligence, he can demonstrate convincingly that, of all the subjects included in the form order marks, the subject which most closely correlates with the form-order, and so *ex hypothesi* with general intelligence, is, of course, mathematics. If, after this argument, there is no widespread urge to abolish form orders and similar statistical operations on marks, then your colleagues deserve all the futile adding, averaging, and ordering of marks they impose on themselves.

But, as no doubt some of you have noticed, I have implicitly admitted in this argument that marks *are* measures of ability, though crude. For example, I spoke of mathematically homogeneous forms and sets determined by mathematical testing and marking. So though my comments on school marks have been adversely critical, I do not condemn marks utterly. They have their uses, but I suggest that we claim too great a precision for them, that we perform many meaningless statistical operations on them, that we waste time and energy on them that might be better spent, and that they help to foster in children an uncritical acceptance of any other numerical data they see in print.

I therefore ought, perhaps, to conclude with some *positive* suggestions for using marks in school. I have only four to make :

1. Mark only with a limited teaching purpose that is explained beforehand to the pupils.
2. Make the marking system as simple as possible, with a ten-point scale at the most.
3. Use letters rather than numbers for marking, to reduce the itch to add them together and find their average.

4. At whatever level you teach, and especially in mathematics, look for qualities other than correctness of technique and answer. Look for imagination, intuition, ingenuity, a sense of judgment and then give these qualities of appreciation as much weight in your marking scheme as you give to technique. It may be difficult, but I think it will be rewarding.

In this way I suggest that school marks could be useful in teaching, that pupils need no longer be misled by the modern forms of number magic that I have described, and that teachers need no longer waste their time being *mere* statisticians.

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1894. He (Johnston) had the rare distinction of batting six times in a test series without once being out. He scored altogether 22 runs and his average, according to the mathematicians, is infinity. It is time Hassett sent him in first to make that division sum easier.—Bruce Harris, *Cricket Triumph*, p. 145. [Per Dr. E. J. F. Primrose.]

1895. ONE FIGURE ACCURACY.

The eyes which had brightened his youth . . . seemed to shine upon him after five-and-thirty years. He remembered such a fair bending neck and clustering hair . . . and now (was) parted from it with a gap of ten thousand long days between.—W. M. Thackeray, *The Newcomes*, ch. XV. [Per Mr. J. Buchanan.

1896. Beyond very much doubt, if the English had permitted themselves a revolution at any time in the last century and a half, the establishment would have been swept away in a paroxysm of reforming zeal, together with the House of Lords, the undecimal currency, the archaic system of weights and measures, the end of the tax year on April 5th., the Lord Mayor of London and a number of other English institutions whose only justification is that it has never seemed to be quite the right time to bring them to an end.—*The Economist*, November, 19, 1955. [Per Mr. J. c. w. De la Bere.]

1897. " What exactly do you mean by magician? "

" Magicians think in an active way ; everyone else thinks passively."

" I see. So mathematicians, philosophers and scientists are magicians? "

" No, people of that sort, if we had them (but we don't) would be recorders. One doesn't need an active mind to record."

" But surely you'd distinguish a man who adds up a column of figures from a man who invents a complex mathematical formula or generalizes about the nature of the universe? "

" There's no magic in a mathematical formula, however complex. It's only a recorder's convenience for his fellow-recorders : it's a part of accountancy, or history. A philosophic concept about the nature of the universe is of the same order : it's part of history."—Robert Graves, *Seven Days in New Crete*, chap. 2. [Per Mr. J. c. w. De la Bere.]

1898. Another feature of patriotic interpretation is the mystical significance assigned to numbers. Numbers seem to have a strange fascination for the human mind ; among logicians from the Greek philosophers onwards, the queer tendency has persisted of treating mathematical propositions, such as $2+2=4$, as the normal type of a true proposition, and we have seen in an earlier chapter how John Locke thought that the evidence for the existence of God was " equal to mathematical certainty." Perhaps the notion of the mystical significance of numbers is not unrelated to this.—A. G. Hebert, S.S.M., *The Authority of the Old Testament*, p. 276. [Per Mr. J. C. W. De la Bere.]

MATHEMATICAL NOTES

2720. " On casting out nines and elevens."

The method of checking arithmetical processes by casting out nines is well known. The check by calculating the remainders on division by eleven (*i.e.* by working the sum as a congruence modulo 11) is less used, because it is more difficult to apply, though it is more powerful than the " nine " test, since it will discover the frequently occurring error of transposition of two adjacent digits.

The remainder when a number is divided by 11 is normally found by subtracting the sum of the digits in the even places, starting from the right, from the sum of the digits in the odd places (since $10 \equiv -1 \pmod{11}$). Subtraction and the need for remembering an intermediate sum can, however, be avoided by taking the complement (mod 11) of the sum of the even place digits and adding this to the sum of the odd place digits. *E.g.*, to test 8193, we may say $9+8=17$, complement 5 ; $5+3+1$ gives remainder 9.

These tests are, of course, only of use where all figures are retained in the working. This condition is most frequently fulfilled in calculations involving money, and for English coinage special rules can be devised for both tests. For casting out nines, since every pound is congruent to 2s. and every shilling to 3d. (mod 9), we have the rule : " Deal with the pounds in the usual way. Double the remainder and add the shillings. Cast off the nines from this result as usual ; multiply the remainder by 3 and add to the pence, casting off nines once again." On account of the subsequent multiplication by 3 we may evidently cast off threes in the pounds and shillings, which, of course, reduces the diagnostic value of the test.

In the case of the 11 test, each £1 is congruent to -2s. and each shilling to $+1$d. Hence we cast out elevens from the pounds as usual, double and take the complement (or take the complement and double, whichever is simpler) and then add the shillings and the pence, dropping elevens on the way. Since the remainder of the pounds column has to be complemented, it may be quicker to proceed as follows : " In the pounds, add the odd place digits, take the complement, and then add the even place digits. Multiply by 2, and add the product to the shillings and pence." *E.g.*, to test £4523 8s. 5d., we say $3+5=8$, complement 3 ; $3+2+4=9$, $2\times 9=18\equiv 7$; $7+8+5=20\equiv 9$.

Since the only multiplication in the 11 test is by 2, which is prime to 11, this test will reveal any error in a single digit, which we have seen not to be the case with the nine test. The 11 test will also reveal interchange of digits in the pounds, but not unfortunately interchange between shillings and pence, nor between pounds and shillings where these differ by a multiple of 11 : *e.g.*, £3 14s. 0d. and £14 3s. 0d. both leave remainder 8d. The 9 test, however, fails whenever in an interchange of this kind the difference is a multiple of 3 : *e.g.*, £8 2s. 0d. and £2 8s. 0d. both leave remainder 0. Evidently the 11 test is much the more powerful of the two.

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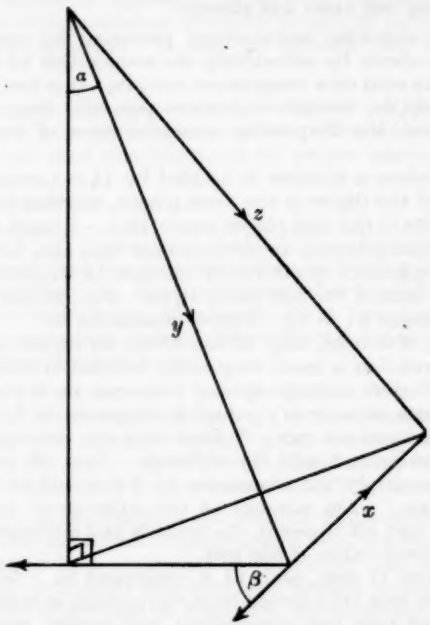
2721. A walk in the rain.

Rain is falling in a north wind at an angle α to the vertical with a velocity y, and a man is to make a journey through the rain in a direction at an angle β with north. At what speed must he walk to keep as dry as possible, assuming the conditions will remain constant?

The amount of rain he encounters, R, will be equal to (time taken) $\times$ (area

exposed normally to direction of rain relative to himself) × (flow of rain relative to himself). We shall take the man to be a sphere of unit radius,



and the flow of rain and the distance to be travelled as units. Let x be the man's speed, and z the velocity of the rain relative to the man. From the diagram we get

$$z=(x^2+2xy\sin\alpha\cos\beta+y^2)^{\frac{1}{2}},$$

and since, at constant density, flow is proportional to velocity, it follows that

$$R=\frac{1}{x}\cdot\pi\cdot\frac{z}{y}$$
$$=\frac{\pi(x^2+2xy\sin\alpha\cos\beta+y^2)^{\frac{1}{2}}}{xy}.$$

Hence
$$\frac{dR}{dx}=\frac{-\pi(x\sin\alpha\cos\beta+y)}{x^2(x^2+2xy\sin\alpha\cos\beta+y^2)^{\frac{1}{2}}}$$
$$=0 \quad \text{when } x=\frac{-y}{\sin\alpha\cos\beta}.$$

Thus when $\beta\leqslant 90°$, the best the man can do is go as fast as possible, but when $\beta>90°$ he must make the southerly component of his speed equal to y . cosec α.

The solution for forms more closely resembling the human form is left to the reader: in these the areas presented by the man will vary with his speed, and a more complex expression for R result. The rule given above, however, is probably a good guide to practice.

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2722. Approximate length of arc of an ellipse.

1. In the recently issued report on the *Teaching of Mathematics in Technical Colleges* reference is made on page 28 to approximations for the length of arc of an ellipse. If this length of arc is written $2\pi E$ (where we may call E the "equivalent radius"), it is pointed out that A, the arithmetic mean of the semi-axes a and b, gives a poor approximation to E (G, the geometric mean, it may be added, is worse), and a better approximation suggested, namely

$$E \simeq \tfrac{1}{2}(3A - G). \quad \text{.....I}$$

It is the purpose of this note to draw attention to another approximation,

$$E \simeq \tfrac{1}{2}(A + R), \quad \text{.....II}$$

where R is the root mean square of a and b.

Actually there is little to choose between these two formulae. For values of b, the semi-minor axis, greater than about $a/10$, I is slightly more accurate, while for the smaller values of b, II is the better. But, especially if the fact that E lies between A and R is known, II is probably the easier to recall.

2. We first consider a function of θ defined by

$$f(\theta) = (a^2 \sin^2\theta + b^2 \cos^2\theta)^{\frac{1}{2}} + (a^2 \cos^2\theta + b^2 \sin^2\theta)^{\frac{1}{2}}, \text{ where } a > b > 0.$$

Now

$$f'(\theta) = \tfrac{1}{2}(a^2 - b^2) \sin 2\theta [(a^2 \sin^2\theta + b^2 \cos^2\theta)^{-\frac{1}{2}} - (a^2 \cos^2\theta + b^2 \sin^2\theta)^{-\frac{1}{2}}],$$

and if $0 < \theta < \frac{1}{4}\pi$, the first factors are obviously positive while the factor in the square brackets is positive if

$$a^2 \sin^2\theta + b^2 \cos^2\theta < a^2 \cos^2\theta + b^2 \sin^2\theta,$$

or, $(a^2 - b^2) \cos 2\theta > 0$, which is so.

Thus for $0 < \theta < \frac{1}{4}\pi$, $f(\theta)$ steadily increases with θ, and thus

$$(a + b) < f(\theta) < (2a^2 + 2b^2)^{\frac{1}{2}}.$$

The length of arc of the ellipse given by $x = a \cos\theta$, $y = b \sin\theta$, is

$$2\pi E = 4\int_0^{\frac{1}{2}\pi} (a^2 \sin^2\theta + b^2 \cos^2\theta)^{\frac{1}{2}}\, d\theta$$

$$= 4\left(\int_0^{\frac{1}{4}\pi} + \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi}\right)(a^2 \sin^2\theta + b^2 \cos^2\theta)^{\frac{1}{2}}\, d\theta,$$

and if in the second integral the appropriate change of variable is made we find

$$2\pi E = 4\int_0^{\frac{1}{4}\pi} f(\theta)\, d\theta.$$

Thus,

$$\pi(a + b) < 2\pi E < \pi(2a^2 + 2b^2)^{\frac{1}{2}},$$

or

$$A < E < R,$$

a result which is interesting in itself.

3. Having straddled the target we now seek an approximation for E in terms of A and R. If e is the eccentricity of the ellipse we have

$$2\pi E = 4a\int_0^{\frac{1}{2}\pi}(1-e^2\cos^2\theta)^{\frac{1}{2}}\,d\theta$$

$$= 4a\int_0^{\frac{1}{2}\pi}\left[1-\tfrac{1}{2}e^2\cos^2\theta-\tfrac{1}{8}e^4\cos^4\theta - \sum_{n=3}^{\infty}\frac{1\,.\,3\,.\,5\,\ldots\,(2n-3)}{2\,.\,4\,.\,6\,\ldots\,(2n)}\cos^{2n}\theta\, e^{2n}\right]d\theta,$$

and when we integrate the cosine terms we obtain the expansion,

$$E = a\left[1-\frac{e^2}{4}-\frac{3e^4}{64}-\sum_{n=3}^{\infty}\frac{1^2\,.\,3^2\,.\,5^2\,.\,\ldots\,.\,(2n-3)^2(2n-1)}{2^2\,.\,4^2\,.\,6^2\,.\,\ldots\,.\,(2n)^2}e^{2n}\right].$$

We also give the expansions for $A=\frac{1}{2}a[1+(1-e^2)^{\frac{1}{2}}]$ and $R=a(1-\frac{1}{2}e^2)^{\frac{1}{2}}$:

$$A = a\left[1-\frac{e^2}{4}-\frac{e^4}{16}-\sum_{n=3}^{\infty}\frac{1}{2}\cdot\frac{1\,.\,3\,.\,5\,\ldots\,(2n-3)}{2\,.\,4\,.\,6\,\ldots\,(2n)}e^{2n}\right],$$

$$R = a\left[1-\frac{e^2}{4}-\frac{e^4}{32}-\sum_{n=3}^{\infty}\frac{1}{2^n}\cdot\frac{1\,.\,3\,.\,5\,\ldots\,(2n-3)}{2\,.\,4\,.\,6\,\ldots\,(2n)}e^{2n}\right].$$

Study of the early terms leads us to take equal weightings of A and R to give $E\simeq\frac{1}{2}(A+R)$, and we find, in fact,

$$E-\tfrac{1}{2}(A+R) = a\sum_{n=0}^{\infty}c_n e^{2n},$$

where $$c_0=c_1=c_2=c_3=0,\quad c_4=5\,.\,2^{-14},$$

and generally for $n>4$,

$$c_n=\frac{1\,.\,3\,.\,5\,\ldots\,(2n-3)}{2\,.\,4\,.\,6\,\ldots\,(2n)}\left[\frac{1}{4}+\frac{1}{2^{n+1}}-\frac{1\,,\,3\,.\,5\,\ldots\,(2n-1)}{2\,.\,4\,.\,6\,\ldots\,(2n)}\right].$$

Thus if e is small the approximation is of order e^6 and further, since for $n\geqslant 5$

$$\frac{1\,.\,3\,.\,5\,\ldots\,(2n-1)}{2\,.\,4\,.\,6\,\ldots\,(2n)}\leqslant\frac{1\,.\,3\,.\,5\,.\,7\,.\,9}{2\,.\,4\,.\,6\,.\,8\,.\,10}=\frac{63}{256}<\frac{1}{4},$$

we conclude that c_n is always positive. Thus $E-\frac{1}{2}(A+R)$ increases with e, and hence the greatest error occurs when $e=1$, which is when the ellipse flattens out to a straight line, the length of arc now being $4a$. Even in this extreme case the error is only about $5{\cdot}2\%$.

4. Similar calculations for the approximation $E\simeq\frac{1}{2}(3A-G)$ give an error term for small e which begins with $-3e^8/2^{14}$, which is even better than what we had with $\frac{1}{2}(A+R)$. But at $e=1$, the percentage error has leapt to some 18%. This calls for an explanation.

We write $$\tfrac{1}{2}(3A-G)=aF(m)$$

where $m=b/a$ and $F(m)=\frac{1}{2}[3(1+m)/2-m^{\frac{1}{2}}]$.

For large values of $m(\simeq 1)$, the graphs of $F(m)$ and E/a closely approach each other, but because of the presence of the term $m^{\frac{1}{2}}$, the graph of $F(m)$ has an infinite gradient at $m=0$, and necessarily departs rapidly from that of E/a for small m.

5. A 5 to 3 weighting of I and II yields

$$E \simeq \tfrac{1}{16}(18A + 3R - 5G), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{III}$$

which turns out to have an error term beginning with $7e^{12}/2^{17}$.

This is luxurious accuracy for small values of e, but in spite of this, for $e = 1$ the formula is disappointing, in fact inferior to II, which is due once again to the presence of a term in G.

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2723. Transport problem.

In Note 2609 (May 1956) the following problem is suggested: "I am at a corner where I have the choice of two independent bus systems, buses on each running at ten-minute intervals. How long may I expect to wait for a bus?"

We assume that the waiting times x, y for the two buses have the rectangular distribution over the interval (0, 10), so that their probability density functions are $f(x) = g(y) = \frac{1}{10}$.

The chance that *neither* comes in the first t minutes is $(1 - t/10)^2$ and so the chance of getting a bus in t minutes is $1 - (1 - t/10)^2$. Hence if $h(u)$ is the density function of $u \equiv \min(x, y)$, we have

$$\int_0^t h(u)\,du = 1 - \left(1 - \frac{t}{10}\right)^2,$$

whence

$$h(u) = 2 \cdot \frac{1}{10} \cdot \left(1 - \frac{u}{10}\right) = \frac{10 - u}{50}$$

so that $E(h) = \int_0^{10} uh(u)\,du = 3\frac{1}{3}$ minutes.

In the case of n independent bus systems we would have

$$\int_0^t h(u)\,du = 1 - \left(1 - \frac{t}{10}\right)^n$$

and therefore

$$h(u) = \frac{n}{10}\left(1 - \frac{u}{10}\right)^{n-1}$$

so that

$$E(u) = \frac{n}{10^n}\int_0^{10} u(10 - u)^{n-1}\,du = 10nB(2, n) = \frac{10}{n+1} \text{ minutes.}$$

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2724. Bibliographical note.

The proposition that the three Apollonian circles of a triangle intersect at angles of 120° [T. A. Honan, this *Gazette*, Vol. 40, No. 333, Oct. 1956, p. 211, Note 2611] is due to E. Lemoine, who in his turn was anticipated by Glaser of Hamburg (*Mathesis*, 1902, p. 147, Q. 1329).

A simple synthetic proof of the proposition may be based on the property that the centre of an Apollonian circle of a triangle is a centre of similitude of the remaining two Apollonian circles (*Mathesis*, Vol. 41, 1927, p. 464, Note 49).

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2725. Oscillating solutions of the duplication formula for the cosine.

In previous Notes (2716, 2717) some solutions of the duplication formula

$$\phi(2x) = 2\phi^2(x) - 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

have been examined. They have all been subject to the condition $\phi(0) = 1$. The formula (1) is also satisfied by $\phi(0) = -\frac{1}{2}$. The questions of the existence and of the nature of solutions with this initial condition naturally arise. It appears that there is an infinite number of such solutions which are continuous at $x = 0$, but that the trivial solution $\phi = \text{const.}$ is the only one which has a derivative at $x = 0$.

We restrict ourselves to positive values of x as no relation has been imposed to relate the values of the function in the two ranges $x \geqslant 0$ and $x \leqslant 0$. Since $\phi(x)$ is continuous at $x = 0$, there will be a number a such that $\phi(x) < 0$ for all x in $0 < x \leqslant a$. The function will be completely determined by its values in the interval $\frac{1}{2}a < x \leqslant a$, since in calculating values at points to the left of this by means of (1) we must always take the negative square root. More precisely, if $\phi(x)$ is negative and $\phi(2x) = \cos 2y$, then $\phi(x)$ is equal to whichever of $\cos y$, $\cos(y + \pi)$ is negative.

Not only is this the case, but a solution of this type can be defined by an arbitrary choice of values in $\frac{1}{2}a < x \leqslant a$; they must, of course, lie in the range $(-1, 0)$. The values calculated from these will lie in the ranges shown in the following table:

Range of x	Greatest possible range of $\phi(x)$
$(\frac{1}{2}a, a)$	$(-1, 0)$
$(\frac{1}{4}a, \frac{1}{2}a)$	$\left(-\sin\frac{\pi}{4}, 0\right)$
$(\frac{1}{8}a, \frac{1}{4}a)$	$\left(-\sin\frac{\pi}{4}, -\sin\frac{\pi}{8}\right)$
$(\frac{1}{16}a, \frac{1}{8}a)$	$\left(-\sin\frac{3\pi}{16}, -\sin\frac{\pi}{8}\right)$
$(\frac{1}{32}a, \frac{1}{16}a)$	$\left(-\sin\frac{3\pi}{16}, -\sin\frac{5\pi}{32}\right)$
....	....

The angles which appear in the second column are the successive partial sums of the series

$$\frac{\pi}{4}\left(1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \ldots\right).$$

As the sum of this series is $\pi/6$, we see that $\phi(x)$ is continuous at $x = 0$.

This calculation also shows that in the interval $(a/2^{n+1}, a/2^n)$ the difference between $\phi(x)$ and $-\frac{1}{2}$ is $O(1/2^n)$. Hence the upper and lower derivatives of $\phi(x)$ at $x = 0$ are both finite. We show next that they are equal in magnitude and opposite in sign. Write $\psi(x) = \phi(x) + \frac{1}{2}$; then

$$\psi(2x) = 2\psi^2(x) - 2\psi(x)$$

and

$$\psi(x) \to 0 \quad \text{as} \quad x \to 0.$$

So

$$\frac{\psi(2x)}{2x} = -\frac{\psi(x)}{x}[1 + o(1)],$$

and it is clear that corresponding to any sequence $\{x_n\}$ which $\to 0$ for which $\psi(x_n)/x_n \to \lambda$, there is another, namely $\{2x_n\}$, for which the limit is $-\lambda$.

It follows that if $\phi'(0)$ exists it must be zero. We have to show that this can occur only when $\phi(x)$ is constant. Let t be such that $-1<\phi(x)<0$ for all x in $0 \leqslant x \leqslant t$ and let $\phi(t) = -\sin\alpha$ where $0<\alpha<\frac{1}{2}\pi$. Then

$$\phi\left(\frac{t}{2}\right) = -\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right),$$

and generally

$$\phi\left(\frac{t}{2^n}\right) = -\sin\left(\frac{\pi}{6}-\frac{(-1)^n\pi}{6\,.\,2^n}+\frac{(-1)^n\alpha}{2^n}\right).$$

Therefore $\phi\left(\frac{t}{2^n}\right)+\frac{1}{2} = \frac{1}{2}\left(1-\cos\frac{6\alpha-\pi}{6\,.\,2^n}\right)+(-1)^n\frac{\sqrt{3}}{2}\sin\frac{6\alpha-\pi}{6\,.\,2^n}.$

Hence $\dfrac{\phi(2^{-n}t)+\frac{1}{2}}{2^{-n}t}$ has limit $\pm\dfrac{(6\alpha-\pi)\sqrt{3}}{12t}$ according as $n\to+\infty$ through even or odd values. Therefore if $\phi'(0)=0$, $\alpha=\pi/6$ for every choice of t, *i.e.*, $\phi(x)$ is constant.

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2726. A problem on strings of beads.

A problem which is of some interest in the theory of semigroups and in other contexts may be stated thus. It is required to construct a string of beads of three colours such that there is no " local repetition " in the pattern, *i.e.*, there do not exist integers $m>0$, n such that the $(n+1)$th to the $(n+m)$th beads have the same respective colours as the $(n+m+1)$th to the $(n+2m)$th. Thus if the beads are represented by the digits 0 1 2 according to their colour, the string must not contain consecutive beads 0 0 of the same colour, or consecutive pairs 0 1 0 1, etc., anywhere in the sequence. The sequence is to extend to infinity in both directions. The only solutions known to me * are indirect to the extent that they involve constructing an auxiliary sequence (of binary and ternary digits respectively) from which the required sequence is derived by a further construction. I give here a construction which is direct, producing the required sequence without any intermediate construction, and which is substantially different from either of the solutions mentioned above. Unlike these latter, it treats the three colours essentially equivalently.

The present solution consists of constructing three finite blocks of beads of equal length, denoted by A, B, C, which are such that given any permissible arrangement of the beads, the corresponding arrangement of the blocks is also permissible. As this arrangement of the blocks gives a longer string of beads than the original one, the process may be repeated to give arbitrarily long strings of beads. If in particular the initial arrangement of the beads is that of one of the blocks, each new string of beads is an extension of the previous one, and this string can therefore be extended without limit.

Consider the blocks

A:	0 1 2 1 0 2 1 2 0 1 2 1 0
B:	1 2 0 2 1 0 2 0 1 2 0 2 1
C:	2 0 1 0 2 1 0 1 2 0 1 0 2

* Marston Morse and Gustav A. Hedlund, *Duke Math. J.* 11 (1944) 1–7; B. J. Birch, unpublished.

It will be noted that these blocks are symmetrical and that a cyclic permutation of 0 1 2 permutes $A\ B\ C$ cyclically. This is not necessary to the solution, but it simplifies the proof by reducing the number of cases to be considered. We show first that no block or pair of different blocks run consecutively has a local repetition with $n \leqslant 7$. To do this we enumerate all possible sequences of digits beginning with 0 which could exhibit such repetition. The underlines show the extent of near approaches to repetition. The choice of the digit 0 is completely general on account of the cyclic relationship between the blocks.

Position of 0		Sequence of digits
A (first)		0 1 2 1 0 2 1 2
A (second)		0 2 1 2 0 1 2 1
A (third)	then B	0 1 2 1 0 1 2 0 2
A (third)	then C	0 1 2 1 0 2 0 1 0
A (last)	then B	0 1 2 0 2 1 0 2
A (last)	then C	0 2 0 1 0 2 1 0 1
B (first)		0 2 1 0 2 0 1 2
B (second)		0 2 0 1 2 0 2 1
B (third)	then A	0 1 2 0 2 1 0 1 2 1
B (third)	then C	0 1 2 0 2 1 2 0 1 0
B (last)	then A	0 2 1 0 1 2 1 0 2 1 2
B (last)	then C	0 2 1 2 0 1 0 2 1 0
C (first)		0 1 0 2 1 0 1 2
C (second)		0 2 1 0 1 2 0 1
C (third)	then A	0 1 2 0 1 0 2 0 1 2 1
C (third)	then B	0 1 2 0 1 0 2 1
C (fourth)	then A	0 1 0 2 0 1 2 1
C (fourth)	then B	0 1 0 2 1 2 0 2
C (last)	then A	0 2 0 1 2 1 0 2 1
C (last)	then B	0 2 1 2 0 2 1 0 2 0

This exhibits that any two different blocks can be fitted together without producing a local repetition of up to seven digits. Further, since these trial sequences are all different in their first eight digits, it follows that if any two sequences of eight or more digits in a string made up of these blocks are identical, then these sequences are similarly placed with respect to the blocks. If in particular two consecutive sequences of more than seven digits are identical, the sequence length must be a multiple of the block length and there must be a local repetition in the block pattern. The construction is therefore successful. We may begin with any block, say A, and let a digit of it, say the middle 1, correspond to A. We then place new blocks on either side of this block corresponding to the digits on either side of the selected 1 (say B for 2 and C for 0), and produce the string indefinitely. This particular string is symmetrical. If we had chosen any digit other than the central 1 it would not have been.

The blocks $A\ B\ C$ were constructed by trial, assuming symmetry and the cyclic relationship. They are the shortest possible such blocks; there is no difficulty in constructing longer ones.

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2727. On Note 2617.

For arbitrary $n>3$ I construct a non-singular $n\times n$ matrix $A=(a_{ij})$ with the properties $\sum_{j=1}^{n} a_{ij}=\sum_{i=1}^{n} a_{ij}=\sum_{i=1}^{n} a_{i\,n-i+1}=1$, whose reciprocal matrix does *not* possess similar properties. This answers in the negative a question raised by the author of Note 2617.

For n odd it suffices to take

$$A=\left(\begin{array}{ccccc|c|c} & & I\left(\frac{n-1}{2}\right) & & & O & O \\ \hline (n-1) & 0 & \dots & 0 & & -(n-2) & O \\ \hline -(n-1) & 0 & \dots & (n-1) & & 0 & \\ 0 & 0 & \dots & 0 & & 0 & I\left(\frac{n-1}{2}\right) \\ \cdot & \cdot & \dots & \cdot & & \cdot & \\ 0 & 0 & \dots & 0 & & 0 & \\ 0 & 0 & \dots & -(n-1) & & (n-1) & \end{array}\right)$$

and for n even

$$A=\left(\begin{array}{ccccc|cc|c} & & I\left(\frac{n-2}{2}\right) & & & O & & O \\ \hline \left(\frac{n-1}{2}\right) & 0 & \dots & 0 & & -\left(\frac{n-3}{2}\right) & 0 & \\ & & & & & & & O \\ \left(\frac{n-3}{2}\right) & 0 & \dots & 0 & & 1 & -\left(\frac{n-3}{2}\right) & \\ \hline -\left(\frac{n-2}{2}\right) & 0 & \dots & \left(\frac{n-2}{2}\right) & & 0 & 0 & \\ 0 & 0 & \dots & 0 & & 0 & 0 & \\ \cdot & \cdot & \dots & \cdot & & \cdot & \cdot & I\left(\frac{n-2}{2}\right) \\ 0 & 0 & \dots & 0 & & 0 & 0 & \\ -\left(\frac{n-2}{2}\right) & 0 & \dots & -\left(\frac{n-2}{2}\right) & & \left(\frac{n-3}{2}\right) & \left(\frac{n-1}{2}\right) & \end{array}\right)$$

University College, Swansea. HAZEL PERFECT

2728. Quadrant dissection.

Let C be any point on a quadrant arc AB, centre O, and let E be the point of trisection of the arc BC, nearer to B.

Points D, G are taken on the arc AC such that arc AD is a fraction x of arc AC and arc AG is a fraction y of AC.

DE meets OG at H. If $x=4y-1$, $\frac{1}{4}\leqslant y\leqslant\frac{1}{3}$, then $EH=OA$.

Interesting particular cases are given by $x=0$, $y=\frac{1}{4}$ and $x=y=\frac{1}{3}$.

A. H. FINLAY

2729. The sum of a certain series involving binomial coefficients.

Many series involving binomial coefficients can be summed by using either of the integral representations (A) and (B), where

$$\binom{x}{n}=\frac{1}{2\pi i}\int_C z^{-n-1}(1+z)^x\,dz, \quad \text{.....(A)}$$

$$\binom{x}{n}=\frac{1}{2\pi i}\int_{C'} z^{-n-1}(1-z)^{-x+n-1}\,dz, \quad \text{.....(B)}$$

x being any complex number, n being any integer and C and C' being the positive paths along any simple closed Jordan curves enclosing $z=0$, where the plane is cut along real z from -1 to $-\infty$ for C and from 1 to ∞ for C' to exclude branch points, the powers taking their principal values here and throughout this note. (A) and (B) are immediate consequences of the residue theorem.

The following result will be proved as an illustration of this and will then be specialised to give the results of Notes 2419, 2559, 2620.

Theorem. If v is real and $v>1$ and u and x are complex then

$$T(u,v,x)\equiv\sum_{s=0}^{\infty}\binom{u+sv}{s}x^s$$

has a radius of convergence given by $r(v)=(v-1)^{v-1}/v^v$ and if $|x|<r(v)$, has the value

$$\frac{1}{(1-v\alpha)(1-\alpha)^u}$$

where α is the only root of $z(1-z)^{v-1}=x$ inside $|z|=1/v$.

The following lemma is required:

Lemma. If $v>1$ is any real and x any complex number such that

$$|x|<(v-1)^{v-1}/v^v,$$

then $z(1-z)^{v-1}=x$ has only one root satisfying $|z|<1/v$, and this root is simple.

Proof. If $|x|<(v-1)^{v-1}/v^v$, $v>1$, then

$$|-x|<|z(1-z)^{v-1}| \text{ when } |z|=1/v,$$

and so by Rouché's theorem $z(1-z)^{v-1}-x$ has the same number of zeros inside $|z|=1/v$ as $z(1-z)^{v-1}$, viz. 1.

Proof of the theorem.

Consider the relation

$$\sum_{s=0}^{\infty}z^{-s-1}(1-z)^{+u+s(1-v)-1}x^s=\frac{(1-z)^{-u+v-s}}{z(1-z)^{v-1}-x}. \quad \text{.....(1)}$$

When $|z|=1/v$ and $|x|<r(v)$, then

$$|xz^{-1}(1-z)^{1-v}|<|x|/r(v)<p<1$$

for some fixed p and so (1) holds and the series is uniformly convergent. It is therefore legitimate to apply $\frac{1}{2\pi i}\int_{C'}dz$ and (*B*) term by term in (1) to give

$$\sum_{s=0}^{\infty} \binom{u+sv}{s} x^s = \frac{1}{2\pi i}\int_{C'} \frac{(1-z)^{-u+v-1}}{z(1-z)^{v-1}-x}\,dz$$

where C' is the positive path along $|z|=1/v$. (The relation (1) is suggested by using (B) to substitute for the binomial coefficients in $T(u, v, x)$ and formally interchanging summation and integration.)

The last contour integral is easily evaluated by the lemma and the residue theorem to show that

$$T(u, v, x) = \frac{1}{(1-v\alpha)(1-\alpha)^u} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

when $|x| < r(v)$ and α is the only root of $z(1-z)^{v-1}=x$ inside $|z|=1/v$.

Radius of Convergence. If $\alpha \to 1/v$ where α takes only real values less than $1/v$, then $x \to r(v)$ along some curve D, say. Since α is unique it follows that $\alpha \to 1/v$ as $x \to r(v)$ along D and hence, by (2), that T is singular when $x = r(v)$. The value $\alpha = 1$ is excluded since $|\alpha| < 1/v < 1$, and so $r(v)$ is the radius of convergence of T, which completes the proof of the theorem.

Applications. When $u=1$, $v=3$ and $x=2/27$, then $r(v)=4/27>x$ and the series $T(u, v, x)$ is convergent. The roots of $z(1-z)^2=2/27$ are $2/3$, $(2-\sqrt{3})/3$, $(2+\sqrt{3})/3$. The only one with modulus less than $1/3$ is $(2-\sqrt{3})/3$, which gives $T(1, 3, 2/27)=3/2$ and proves Mr. D. F. Ferguson's result (Note 2419)

$$\sum_{r=1}^{\infty} \binom{1+3r}{r} \left(\frac{2}{27}\right)^r = \frac{1}{2},$$

which has been proved by Prof. G. N. Watson (Note 2559).

If the theorem is used to sum the c series, $w_r^{-b}T(u, v, xw_r)$, $v>1$, $r=1, \ldots c$, where b and c are integers and w_r are the c roots of unity, and the sums are added, it is possible to express

$$\sum_{k=0}^{\infty} \binom{n+ak}{b+ck} x^{b+ck}$$

in terms of the c roots of $z^c(1-z)^{a-c}=x^c$ inside $|z|=c/a$ for $a>c$, complex n and real a.

Thus if we write

$$\sum_{k=1}^{\infty} \binom{4k}{2k} x^{2k} = -1 + \tfrac{1}{2}[T(0, 2, x) + T(0, 2, -x)],$$

then the theorem shows that $|x| = \frac{1}{4}$ is the common radius of convergence of the last two series. When $x=6/25$ it follows, by taking the roots of the quadratics $z(1-z) = \pm 6/25$ with moduli less than $\frac{1}{2}$, that $T(0, 2, x)=5$ and $T(0, 2, -x)=5/7$. This gives

$$\sum_{k=1}^{\infty} \binom{4k}{2k} x^{2k} = \frac{13}{7},$$

which proves a result introduced by Mr. D. G. Tahta (Note 2620).

Extensions. It is of interest to know the value of $T(u, v, x)$ when v is not a real number satisfying $v>1$.

When $v<0$ the transformation

$$\binom{u+sv}{s} x^s = \binom{-u-1+(1-v)s}{s} (-x)^s \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

gives a new coefficient of s which enables the theorem to be applied.

When $0<v<1$, the analysis given here requires modification. Instead of $|z|=1/v$ a circle $|z|=R>1$ may be taken as contour except that the cut $(z=1, \infty)$ is avoided by a U-shaped detour. By taking R sufficiently large and the detour close enough to the cut, it is possible to show that the series in (1) is uniformly convergent when $|x|<|r(v)|$ and may be integrated about this contour. It can also be shown that $z(1-z)^{v-1}=x$ has only one root β say, when $|x|<|r(v)|$ (the power must always be given its principal value). The proof is the same as that of our lemma, except that the new contour is taken instead of $|z|=1/v$. T is then given by

$$T(u, v, x)=\frac{1}{(1-v\beta)(1-\beta)^u}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

When v is complex the principle of analytic continuation shows that $T(u, v, x)$ is still given by (2) or (6). But it is not possible, in general, to specify the required root of $z(1-z)^{v-1}=x$ as the only root in some simple circle for all x inside the circle of convergence of T, *i.e.*, $|x|<|r(v)|$. However, when $v=1+it$ where t is real, the above analysis is easily modified by taking C' as the circle $|z|=1/\sqrt{1+t^2}$ to show that the required root is the only root of $z(1-z)^{it}=x$ inside $|z|=1/\sqrt{1+t^2}$. The use of (5) enables this last result to be applied when v is purely imaginary.

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2730. Some properties of the triangle in cartesian co-ordinates.

In the *Gazette* for July, 1941 (Vol. XXV, No. 265, pp. 150–5), some results are obtained for the triangle ABC in which the rectangular cartesian co-ordinates of A, B, C referred to axes through the circumcentre O are

$$(R\cos\theta_i, R\sin\theta_i), \quad i=1, 2, 3,$$

with the condition that $0\leqslant\theta_1<\theta_2<\theta_3<2\pi$

The angles of the triangle are then given by

$$A=\frac{\theta_3-\theta_2}{2}, \quad B=\pi-\frac{\theta_3-\theta_1}{2}, \quad C=\frac{\theta_2-\theta_1}{2}.$$

Hence with the usual notation $a=R\sin\dfrac{\theta_3-\theta_2}{2}$, etc.

Also the area of the triangle is given by

$$\begin{aligned}\Delta&=\tfrac{1}{2}R^2(\sin 2A+\sin 2B+\sin 2C)\\&=\tfrac{1}{2}R^2[\sin(\theta_3-\theta_2)-\sin(\theta_3-\theta_1)+\sin(\theta_2-\theta_1)],\end{aligned}$$

or by $$\Delta=\tfrac{1}{2}bc\sin A=2R^2\sin\frac{\theta_3-\theta_2}{2}\sin\frac{\theta_3-\theta_1}{2}\sin\frac{\theta_2-\theta_1}{2},$$

and these two expressions can easily be shown to be equivalent.

The semi-perimeter s is given by

$$s=R\left(\sin\frac{\theta_3-\theta_2}{2}+\sin\frac{\theta_3-\theta_1}{2}+\sin\frac{\theta_2-\theta_1}{2}\right),$$

which can be reduced by conventional manipulation to

$$s=4R\cos\frac{\theta_3-\theta_2}{4}\sin\frac{\theta_3-\theta_1}{4}\cos\frac{\theta_2-\theta_1}{4},$$

whence
$$r = \triangle/s = 4R \sin\frac{\theta_3-\theta_2}{4}\cos\frac{\theta_3-\theta_1}{4}\sin\frac{\theta_2-\theta_1}{4}$$
$$= 2R\cos\frac{\theta_2-\theta_1}{4}\left(\cos\frac{\theta_3-2\theta_2+\theta_1}{4} - \cos\frac{\theta_3-\theta_1}{4}\right)$$
$$= R\left(\cos\frac{\theta_3-\theta_2}{2} - \cos\frac{\theta_3-\theta_1}{2} + \cos\frac{\theta_2-\theta_1}{2} - 1\right), \quad \ldots\ldots\ldots\ldots\ldots(1)$$

an alternative form which is sometimes useful.

Similarly we may obtain the expression

$$s-a = 4R\cos\frac{\theta_3-\theta_2}{4}\cos\frac{\theta_3-\theta_1}{4}\sin\frac{\theta_2-\theta_1}{4},$$

and others for $s-b$ and $s-c$, with corresponding values for r_1, r_2, r_3.

The internal bisectors of the angles of the triangle ABC meet at the incentre I and cut the circumference again at points the angular co-ordinates of which are $\frac{1}{2}(\theta_2+\theta_3)$, $\pi+\frac{1}{2}(\theta_3+\theta_1)$, $\frac{1}{2}(\theta_1+\theta_2)$. Calling these points A_1, B_1, C_1, we can easily show that AA_1 is perpendicular to B_1C_1, etc., so that I is the orthocentre of the triangle $A_1B_1C_1$.

Now in the previous article it was proved that the orthocentre H of triangle ABC has co-ordinates

$$[R(\cos\theta_1+\cos\theta_2+\cos\theta_3),\quad R(\sin\theta_1+\sin\theta_2+\sin\theta_3)].$$

Hence the co-ordinates of I are

$$\left\{R\left(\cos\frac{\theta_2+\theta_3}{2} - \cos\frac{\theta_3+\theta_1}{2} + \cos\frac{\theta_1+\theta_2}{2}\right),\right.$$
$$\left. R\left(\sin\frac{\theta_2+\theta_3}{2} - \sin\frac{\theta_3+\theta_1}{2} + \sin\frac{\theta_1+\theta_2}{2}\right)\right\}.$$

Similar expressions can be written down for the co-ordinates of the ex-centres I_1, I_2, I_3 by changing the signs of two terms in each bracket in cyclic order.

As a simple application we may take the following:

$$OI^2 = R^2\left\{\left(\cos\frac{\theta_2+\theta_3}{2} - \cos\frac{\theta_3+\theta_1}{2} + \cos\frac{\theta_1+\theta_2}{2}\right)^2\right.$$
$$\left. + \left(\sin\frac{\theta_2+\theta_3}{2} - \sin\frac{\theta_3+\theta_1}{2} + \sin\frac{\theta_1+\theta_2}{2}\right)^2\right\}$$
$$= R^2\left\{3 - 2\cos\frac{\theta_3-\theta_2}{2} + 2\cos\frac{\theta_3-\theta_1}{2} - 2\cos\frac{\theta_2-\theta_1}{2}\right\}$$
$$= R^2 - 2Rr, \quad \text{by virtue of equation (1).} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

The previous article also showed that the nine-point centre N of the triangle ABC has co-ordinates

$$\left\{\frac{R}{2}(\cos\theta_1+\cos\theta_2+\cos\theta_3),\quad \frac{R}{2}(\sin\theta_1+\sin\theta_2+\sin\theta_3)\right\},$$

which can be written

$$\left[\frac{R}{4}\left\{(\cos\theta_2+\cos\theta_3)+(\cos\theta_3+\cos\theta_1)+(\cos\theta_1+\cos\theta_2)\right\},\right.$$
$$\left.\frac{R}{4}\left\{(\sin\theta_2+\sin\theta_3)+(\sin\theta_3+\sin\theta_1)+(\sin\theta_1+\sin\theta_2)\right\}\right];$$

or

$$\left\{\frac{R}{2}\left(\cos\frac{\theta_2+\theta_3}{2}\cos\frac{\theta_3-\theta_2}{2}+\cos\frac{\theta_3+\theta_1}{2}\cos\frac{\theta_3-\theta_1}{2}+\cos\frac{\theta_1+\theta_2}{2}\cos\frac{\theta_3-\theta_1}{2}\right),\right.$$
$$\left.\frac{R}{2}\left(\sin\frac{\theta_2+\theta_3}{2}\cos\frac{\theta_3-\theta_2}{2}+\sin\frac{\theta_3+\theta_1}{2}\cos\frac{\theta_3-\theta_1}{2}+\sin\frac{\theta_1+\theta_2}{2}\cos\frac{\theta_3-\theta_1}{2}\right)\right\}.$$

We therefore have

$$\frac{NI^2}{R^2}=\left\{\cos\frac{\theta_2+\theta_3}{2}\left(1-\tfrac{1}{2}\cos\frac{\theta_3-\theta_2}{2}\right)-\cos\frac{\theta_3+\theta_1}{2}\left(1+\tfrac{1}{2}\cos\frac{\theta_3-\theta_1}{2}\right)\right.$$
$$\left.+\cos\frac{\theta_1+\theta_2}{2}\left(1-\tfrac{1}{2}\cos\frac{\theta_3-\theta_1}{2}\right)\right\}^2$$
$$+\left\{\sin\frac{\theta_2+\theta_3}{2}\left(1-\tfrac{1}{2}\cos\frac{\theta_3-\theta_2}{2}\right)-\sin\frac{\theta_3+\theta_1}{2}\left(1+\tfrac{1}{2}\cos\frac{\theta_3-\theta_1}{2}\right)\right.$$
$$\left.+\sin\frac{\theta_1+\theta_2}{2}\left(1-\tfrac{1}{2}\cos\frac{\theta_3-\theta_1}{2}\right)\right\}^2.$$

On squaring out and combining corresponding pairs of terms we have

$$\frac{NI^2}{R^2}=\left(1-\tfrac{1}{2}\cos\frac{\theta_3-\theta_2}{2}\right)^2+\left(1+\tfrac{1}{2}\cos\frac{\theta_3-\theta_1}{2}\right)^2+\left(1-\tfrac{1}{2}\cos\frac{\theta_2-\theta_1}{2}\right)^2$$
$$-2\cos\frac{\theta_3-\theta_2}{2}\left(1+\tfrac{1}{2}\cos\frac{\theta_3-\theta_1}{2}\right)\left(1-\tfrac{1}{2}\cos\frac{\theta_2-\theta_1}{2}\right)$$
$$+2\cos\frac{\theta_2-\theta_1}{2}\left(1-\tfrac{1}{2}\cos\frac{\theta_2-\theta_1}{2}\right)\left(1-\tfrac{1}{2}\cos\frac{\theta_3-\theta_2}{2}\right)$$
$$-2\cos\frac{\theta_3-\theta_1}{2}\left(1-\tfrac{1}{2}\cos\frac{\theta_3-\theta_2}{2}\right)\left(1+\tfrac{1}{2}\cos\frac{\theta_3-\theta_1}{2}\right).$$

The only further manipulation needed is to expand the term

$$\frac{3}{2}\cos\frac{\theta_3-\theta_2}{2}\cos\frac{\theta_3-\theta_1}{2}\cos\frac{\theta_2-\theta_1}{2}$$

into the form $\dfrac{3}{4}\left(\cos^2\dfrac{\theta_3-\theta_2}{2}+\cos^2\dfrac{\theta_3-\theta_1}{2}+\cos^2\dfrac{\theta_2-\theta_1}{2}-1\right)$,

and the whole expression can at once be rewritten as

$$\frac{NI^2}{R^2}=\left\{\frac{3}{2}-\left(\cos\frac{\theta_3-\theta_2}{2}-\cos\frac{\theta_3-\theta_1}{2}+\cos\frac{\theta_2-\theta_1}{2}\right)\right\}^2$$
$$=\left(\frac{1}{2}-\frac{r}{R}\right)^2, \quad \text{by virtue of equation (1).}$$

$$NI=\frac{R}{2}-r, \quad \text{since equation (2) shows this to be positive.}$$

As $R/2$ is the radius of the nine-point circle, we have proved Feuerbach's Theorem that the nine-point circle has internal contact with the incircle. Changes of sign are the only alterations needed to prove the corresponding result for the nine-point circle and the escribed circles.

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2731. Linear recurrence relations.

1. Mr. Max Rumney, investigating sequences u_r generated by the relation $u_{n+2}=pu_{n+1}-qu_n$ for particular numerical values of p, q, u_0 and u_1, noticed that u_{2n} and u_{3n} could be expressed as homogeneous polynomials in u_{n+1} and u_n of the second and third degrees respectively, whose coefficients were independent of n. If $u_{2n}=au_{n+1}^2+bu_{n+1}u_n+cu_n^2$, then by substituting pu_n-qu_{n-1} for u_{n+1} we have also $u_{2n}=a'u_n^2+b'u_nu_{n-1}+c'u_{n-1}^2$, and again

$$u_{2n}=a''u_{n-1}^2+b''u_{n-1}u_{n-2}+c''u_{n-2}^2.$$

Mr. Rumney discovered that the determinant $|\,a\;b'\;c''\,|=0$ and that the fourth order determinant got by proceeding similarly with u_{3n} also vanished. The purpose of this note is to give a proof of Mr. Rumney's interesting discoveries.

1a. Illustration.

$$u_{n+2}=2u_{n+1}-3u_n,\quad u_0=1,\quad u_1=2$$

gives the sequence

$$r=\ldots\ -2\quad -1\quad 0\quad 1\quad 2\quad 3\quad 4\quad 5\quad 6\ \ldots,$$

$$u_r=\ldots\ -\tfrac{1}{3}\quad 0\quad 1\quad 2\quad 1\quad -4\quad -11\quad -10\quad 13\ \ldots$$

It can easily be verified that

$$\begin{aligned}u_{2n}&=-\tfrac{1}{3}u_{n+1}^2+\tfrac{4}{3}u_{n+1}u_n-\tfrac{1}{3}u_n\\&=u_n^2-3u_{n-1}^2\\&=u_{n-1}^2-12u_{n-1}u_{n-2}+9u_{n-2}^2,\end{aligned}$$

and the determinant $\begin{vmatrix}-\frac{1}{3} & \frac{4}{3} & -\frac{1}{3}\\ 1 & 0 & -3\\ 1 & -12 & 9\end{vmatrix}=0.$

Also
$$\begin{aligned}u_{3n}&=-\tfrac{2}{9}u_{n+1}^3+\tfrac{1}{3}u_{n+1}^2u_n+\tfrac{4}{3}u_{n+1}u_n^2-\tfrac{11}{9}u_n^3\\&=u_n^3-6u_n^2u_{n-1}-9u_nu_{n-1}^2+6u_{n-1}^3\\&=-4u_{n-1}^3+51u_{n-1}^2u_{n-2}+54u_{n-1}u_{n-2}^2-27u_{n-2}^3\\&=13u_{n-2}^3-348u_{n-2}^2u_{n-3}-297u_{n-2}u_{n-3}^2+108u_{n-3}^3\end{aligned}$$

and the determinant $|\,-\tfrac{2}{9},\ -6,\ 54,\ 108\,|=0.$

Mr. Rumney also conjectured correctly that similar results held for u_{tn}.

In fact $-3^{t-1}u_{tn}=\sum_{r=0}^{t}\binom{t}{r}u_{t+r-2}u_{n+1}^{t-r}(-u_n)^r$, for this sequence.

2. We assume $q(p^2-4q)\neq 0$, so that the equation $x^2-px+q=0$ has two non-zero distinct roots α and β. Then it is well known that $u_n=A\alpha^n+B\beta^n$ where A and B are constants determined by $A+B=u_0$ and $A\alpha+B\beta=u_1$. (Note that n may be a negative integer and the sequence extended to the left.)

From $u_n=A\alpha^n+B\beta^n$ and $u_{n+1}=A\alpha^{n+1}+B\beta^{n+1}$ it follows that

$$u_{n+1}-\alpha u_n=B\beta^n(\beta-\alpha)\quad\text{and}\quad u_{n+1}-\beta u_n=A\alpha^n(\alpha-\beta).$$

We define

$$k_n = -(u_{n+1} - \alpha u_n)(u_{n+1} - \beta u_n) = AB\alpha^n\beta^n(\alpha - \beta)^2 ;$$

then $$k_n = -(u_{n+1}^2 - pu_{n+1}u_n + qu_n^2) \quad = u_{n+2}u_n - u_{n+1}^2,$$

and $$k_n = q^n k_0.$$

We now express u_{2n-r} as a polynomial in u_{n+1} and u_n.

$u_{2n-r} = A\alpha^{2n-r} + B\beta^{2n-r}$. Multiplying each side by $AB\alpha^r\beta^r(\alpha - \beta)^2$,

$$k_r u_{2n-r} = B\beta^r[A^2\alpha^{2n}(\alpha - \beta)^2] + A\alpha^r[B^2\beta^{2n}(\beta - \alpha)^2]$$

$$= B\beta^r(u_{n+1} - \beta u_n)^2 + A\alpha^r(u_{n+1} - \alpha u_n)^2,$$

whence $$k_r u_{2n-r} = u_r u_{n+1}^2 - 2u_{r+1}u_{n+1}u_n + u_{r+2}u_n^2.*$$

Taking particular cases,

$$k_0 u_{2m} = u_0 u_{m+1}^2 - 2u_1 u_{m+1}u_m + u_2 u_m^2,$$

$$k_{-2} u_{2m} = u_{-2} u_m^2 - 2u_{-1}u_m u_{m-1} + u_0 u_{m-1}^2,$$

$$k_{-4} u_{2m} = u_{-4} u_{m-1}^2 - 2u_{-3}u_{m-1}u_{m-2} + u_{-2} u_{m-2}^2.$$

The determinant of coefficients vanishes because the original relation

$$u_{n+2} - pu_{n-1} + qu_n = 0$$

makes column (iii) $+ \frac{1}{2}p$. column (ii) $+ q$. column (i) $= 0$.

Corresponding coefficients in three consecutive expressions for u_{2n}, which are (a'', a', a), (b'', b', b) and (c'', c', c) in the notation of the first paragraph, satisfy $v_{n+2} - (p^2 - 2q)v_{n+1} + q^2 v_n = 0$, the relation whose roots are the squares of the roots of the original relation.

3. For $t > 2$ the procedure for expressing u_{tn-r} in terms of u_{n+1} and u_n is similar, but the coefficients are not so simple.

$$u_{tn-r} = A\alpha^{tn-r} + B\beta^{tn-r}.$$

$$\alpha^r\beta^r(\alpha - \beta)^t u_{tn-r} = A^{1-t}\beta^r[A\alpha^n(\alpha - \beta)]^t + (-)^t B^{1-t}\alpha^r[B\beta^n(\beta - \alpha)]^t$$

$$= A^{1-t}\beta^r[u_{n+1} - \beta u_n]^t + (-)^t B^{1-t}\alpha^r[u_{n+1} - \alpha u_n]^t.$$

Define c_s as $(\alpha - \beta)^{-t}[A^{1-t}\beta^s + (-)^t B^{1-t}\alpha^s]$. Then

$$q^r u_{tn-r} = c_r u_{n+1}^t - \binom{t}{1} c_{r+1} u_{n+1}^{t-1} u_n + \binom{t}{2} c_{r+2} u_{n+1}^{t-2} u_n^2 + \ldots + c_{r+t}(-u_n)^t.$$

Since c_s is of the form $P\alpha^s + Q\beta^s$ it satisfies $u_{n+2} - pu_{n+1} + qu_n = 0$ and, as before, the determinant of coefficients will vanish, because every third order determinant in it vanishes.

4. With a linear recurrence relation of the kth order (having distinct roots) it is possible to express u_{tn+r} as a homogeneous polynomial of the tth degree in $u_{n+k-1}, u_{n+k-2}, \ldots u_n$. We illustrate with a fourth order relation

$$u_{n+4} + a_1 u_{n+3} + a_2 u_{n+2} + a_3 u_{n+1} + a_4 u_n = 0.$$

* By putting $k_r = u_{r+2}u_r - u_{r+1}^2$ this can be written

$$(u_{2n-r}u_r - u_n^2)(u_{r+2}u_r - u_{r+1}^2) = (u_{n+1}u_r - u_n u_{r+1})^2.$$

Let $P(x) \equiv x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 \equiv \prod_{s=1}^{4} (x - \alpha_s) \equiv (x - \alpha_s)(x^3 + p_sx^2 + q_sx + r_s)$,

$$u_n = A_1\alpha_1^n + A_2\alpha_2^n + A_3\alpha_3^n + A_4\alpha_4^n$$

where no two of α_s are equal ; and

$$u_{n+3} + p_1u_{n+2} + q_1u_{n+1} + r_1u_n = A_1\alpha_1^n(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_1 - \alpha_4) = A_1\alpha_1^nP'(\alpha_1)$$

$$u_{tn+r} = \Sigma A_1\alpha_1^{tn+r} = \Sigma A_1^{1-t}\alpha_1^r[P'(\alpha_1)]^{-t}(u_{n+3} + p_1u_{n+2} + q_1u_{n+1} + r_1u_n)^t.$$

Thus u_{tn+r} is a homogeneous polynomial in u_{n+3}, u_{n+2}, u_{n+1}, u_n of the tth degree whose coefficients are of the form $\sum_{s=1}^{4} B_s\alpha_s^r$ where B_s are independent of n and r.

Corresponding coefficients in u_{tn+r} for five successive values of r satisfy the original recurrence relation. Corresponding coefficients in u_{tn} for five successive values of n satisfy a recurrence relation whose roots are the tth powers of the roots of the original relation.

In the general case the calculation of the $\binom{t+k-1}{k-1}$ coefficients is a matter of some labour. When $k>2$, if the roots of the relation cannot be got easily it seems best to solve $\binom{t+k-1}{k-1}$ linear simultaneous equations got by taking particular values of n.

4a. Illustration.

With $u_{n+3} = u_{n+2} + u_{n+1} + u_n$, $u_0 = u_1 = 0$, $u_2 = 1$, we have

$$u_{2n+r} = u_ru_{n+2}^2 + v_ru_{n+1}^2 + u_{r-2}u_n^2 + 2w_{r-2}u_{n+1}u_n + 2u_{r-1}u_nu_{n+2} + 2w_{r-1}u_{n+2}u_{n+1},$$

where v_r and w_r satisfy the original recurrence relation, and

$$v_0 = v_2 = w_0 = w_2 = 1 ; \quad v_1 = -1, \quad w_1 = 0.$$

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2732. On relatively prime sequences.

If $P(x)$ is a polynomial with integral coefficients, then so is its k-th iterate, defined recursively by

$$P^0(x) = x, \quad P^{k+1}(x) = P(P^k(x)).$$

If a and b are integers, then

$$P^k(a) \equiv P^k(b) \pmod{(a-b)}.$$

In particular, for $a = P^n(c)$ and $b = 0$, we have

$$P^{k+n}(c) \equiv P^k(0) \pmod{P^n(c)}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

We shall call a sequence $\{a_n\}_{n\geq 0}$ *relatively prime* if $(a_m, a_n) = 1$ for all m and n with $m \neq n$. From (1) we obtain

THEOREM 1. *$\{P^n(c)\}_{n\geq 0}$ is relatively prime if and only if $(P^k(0), P^n(c)) = 1$ for all $k \geq 1$, $n \geq 0$.*

From this follows immediately a result by Bellman :* If $P^k(0) = P(0) \neq 1$ for $k \geq 1$, and if $(a, P(0)) = 1$ implies $(P(a), P(0)) = 1$, then $\{P^n(c)\}_{n\geq 0}$ is relatively prime whenever $(c, P(0)) = 1$.

* Richard Bellman, A note on relatively prime sequences, Bull. Amer. Math. Soc. 53 (1947) pp. 778–779.

We shall now construct all polynomials $P(x)$ for which $\{P^n(c)\}_{n\geqslant 0}$ is relatively prime for all c. According to theorem 1, this happens if and only if $P^k(0)=\pm 1$ for all $k\geqslant 1$, as is seen by taking $n=k$ and $c=0$. But then $\{P^k(0)\}_{k\geqslant 0}$ must be equal to one of the following six sequences:

$$\{0, 1, 1, \dots\}; \quad \{0, 1, -1, 1, \dots\}; \quad \{0, 1, -1, -1, \dots\};$$

$$\{0, -1, 1, 1, \dots\}; \quad \{0, -1, 1, -1, \dots\}; \quad \{0, -1, -1, \dots\},$$

corresponding to the cases $P(0)=P(1)=1$; $P(0)=P(-1)=1$, $P(1)=-1$; $P(0)=1, P(\pm 1)=-1$; $P(0)=-1, P(\pm 1)=1$; $P(0)=P(1)=-1, P(-1)=1$; $P(0)=-1, P(-1)=-1$.

It is easily seen that the general solution $P(x)$ (with integral coefficients) of m equations

$$P(a_k)=a_{k+1}, \quad k=0, 1, \dots m-1,$$

is obtained from a particular solution $P_1(x)$ as follows:

$$P(x)=P_1(x)+(x-a_1)\dots(x-a_{m-1})Q(x), \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(2)$$

where $Q(x)$ is any polynomial with integral coefficients.

THEOREM 2. *$\{P^n(c)\}_{n\geqslant 0}$ is relatively prime for all c if and only if $P(x)$ belongs to one of the six classes of polynomials*

$$1+x(x-1)Q(x); \quad 1-x-x^2+x(x^2-1)Q(x); \quad 1-2x^2+x(x^2-1)Q(x);$$

$$2x^2-1+x(x^2-1)Q(x); \quad x^2-x-1+x(x^2-1)Q(x); \quad -1+x(x+1)Q(x).$$

Proof. In view of (2) we need only verify that the particular solutions yield the six sequences given above.

McGill University. J. LAMBEK
University of Alberta. L. MOSER

2733. On sums of two squares.

The purpose of this note is to prove the following

THEOREM. *An integer is the sum of two squares if and only if it may be expressed in the form $2^n(4N+1)$, where n is any integer and N is the sum of two triangular numbers.*

Suppose first that N is the sum of the triangular numbers $\frac{1}{2}a(a+1)$ and $\frac{1}{2}b(b+1)$. Then

$$4N+1=(a-b)^2+(a+b+1)^2 \quad \text{and} \quad 2(4N+1)=(2a+1)^2+(2b+1)^2,$$

and therefore $2^n(4N+1)$ is a sum of 2 squares for any n.

For the converse, let $m=x^2+y^2$; we consider separately the cases which arise when both, one or none of x, y is odd.

(i) $x=2a+1$, $y=2b+1$.

Then
$$m=2[(a-b)^2+(a+b+1)^2]$$

and of $a-b$, $a+b+1$ one must be odd and the other even.

(ii) $x=2^p(2a+1)$, $y=2^q(2b+1)$ whese $1\leqslant p\leqslant q$; then, if $p<q$,

$$m=2^{2p}[(2a+1)^2+2^{2(q-p)}(2b+1)^2];$$

and if $p=q$,

$$m=2^{2p}[(2a+1)^2+(2b+1)^2]=2^{2p+1}[(a-b)^2+(a+b+1)^2].$$

(iii) $x=2a$, $y=2b+1$, then

$$m=4(a^2+b^2+b)+1,$$

so that m is of the form $4N+1$.

Thus in all three cases m is of the form $2^n(4N+1)$, where $4N+1$ is the sum of an even and an odd square. It remains to prove that if $4N+1$ is a sum of two squares, then N is a sum of two triangular numbers.

Let $4N+1=(2a)^2+(2b+1)^2$ so that $N=a^2+b^2+b$; then

$$N=\tfrac{1}{2}(a+b)(a+b+1)+\tfrac{1}{2}(a-b-1)(a-b) \quad \text{.....(iv)}$$
$$=\tfrac{1}{2}(a+b)(a+b+1)+\tfrac{1}{2}(b-a)(b-a+1). \quad \text{.....(v)}$$

If $a>b$, then (iv) expresses N as the sum of two triangular numbers, and if $a\leqslant b$, then N is so expressed by (v), which completes the proof.

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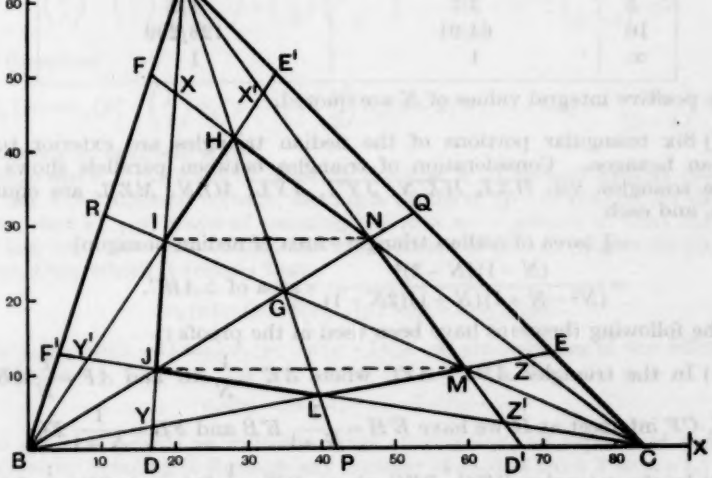
2734. Nedians and the nedian hexagon.

In the triangle ABC (see Fig.) AP, BQ, CR are the medians. Their common intersection is at G, the centroid of the triangle. AD, BE, CF and AD', BE', CF' are the forward and backward *nedians* respectively for the same value of N. In the figure N is equal to five. The $\triangle$s XYZ, $X'Y'Z'$ are the forward and backward nedian triangles respectively. Their common portion is the nedian hexagon $HIJLMN$.



The following statements have been proved to be true (this *Gazette*, No. 332, p. 109):

(1) The points A, H, G, L, P are collinear. Similarly B, J, G, N, Q and C, M, G, I, R.

(2) The lines FE', XX', IN, JM, FE', $Y'Z$, YZ' are parallel to BC. Similar sets of lines are parallel to CA and AB respectively.

(3) The point H, the mid-point of the lines IN, JM and the point L lie on the median AP. Similar statements hold for the other medians.

(4) The median hexagon may be supposed made up of the ΔHIN, the trapezium $IJMN$ and the ΔLJM. From (3) it follows that the centre of gravity of the nedian hexagon lies on the median AP. Similarly it lies on the other medians. Therefore the centre of gravity of the nedian hexagon is coincident with G, the centroid of the triangle ABC. As stated in the previous article, the centres of gravity of the nedian triangles are also at G.

(5) The area of the nedian hexagon

$$= \frac{2(N-2)^2}{(N+1)(2N-1)} \times \text{area of } \Delta ABC.$$

One proof of this consists in finding the coordinates of H, I, J, L, M, N in terms of N, the base a and the altitude h of the ΔABC and using the ordinary formulae for areas (see also Note 2593).

(6) As stated in the previous communication the area of each nedian triangle

$$= \frac{(N-2)^2}{N^2-N+1} \times \text{area of } \Delta ABC;$$

and now the following table can be made:

N	$\frac{\text{Area of nedian } \Delta}{\text{Area of } \Delta ABC}$	$\frac{\text{Area of nedian hexagon}}{\text{Area of } \Delta ABC}$
2	0	0
3	1/7	1/10
5	3/7	1/3
10	64/91	128/209
∞	1	1

Only positive integral values of N are quoted.

(7) Six triangular portions of the nedian triangles are exterior to the nedian hexagon. Consideration of triangles between parallels shows that these triangles, viz. HXI, $HX'N$, $JY'I$, JYL, MZN, $MZ'L$ are equal in area, and each

$$= \tfrac{1}{6} \text{ (area of nedian triangle} - \text{area of nedian hexagon)}$$

$$= \frac{(N-1)(N-2)^2}{(N^2-N+1)(N+1)(2N-1)} \times \text{area of } \Delta ABC.$$

The following theorems have been used in the proofs:

(1) In the triangles ABE', AFC where $AE' = \frac{1}{N} AC$ and $AF = \frac{1}{N} AB$ and BE', CF intersect at H we have $E'H = \frac{1}{N+1} E'B$ and $FH = \frac{1}{N+1} FC$.

(2) In the triangles $BF'C$, BEC where $BF' = \frac{1}{N} BA$ and $CE = \frac{1}{N} CA$ and BE, CF' intersect at L, we have $\frac{EL}{LB} = \frac{N-1}{N}$.

(3) As stated in the previous communication, the ratios

$$AX : XY : XD = N : N(N-2) : 1,$$

with the same ratios for the other nedians.

In the accompanying figure $N=5$, and the triangle ABC has been chosen of such shape to give simple values to the coordinates of the chief points. With origin at B, axis of x along BC and axis of $y \perp BC$, the coordinates are as follows :

$A(21, 63)$, $B(0, 0)$, $C(84, 0)$.
$D(16{\cdot}8, 0)$, $D'(67{\cdot}2, 0)$, $E(71{\cdot}4, 12{\cdot}6)$, $E'(33{\cdot}6, 50{\cdot}4)$, $F(16{\cdot}8, 50{\cdot}4)$, $F'(4{\cdot}2, 12{\cdot}6)$.
$X(20, 48)$, $Y(17, 3)$, $Z(68, 12)$.
$X'(32, 48)$, $Y'(8, 12)$, $Z'(65, 3)$.
$H(28, 42)$, $I(18\frac{2}{3}, 28)$, $J(17\frac{1}{2}, 10\frac{1}{2})$, $L(39\frac{2}{3}, 7)$, $M(59\frac{1}{2}, 10\frac{1}{2})$, $N(46\frac{2}{3}, 28)$.

Physics Department, University of Toronto. JOHN SATTERLY

2735. Colouring sectors.

In how many ways can a disc divided into n sectors be painted, using *exactly* p colours, so that no two adjacent sectors are painted the same colour? Denote the number of ways by $u_n(p)$, and let $v_n(p)$ be the number of ways in which the disc may be painted using *not more than* p colours. Then

$$v_n(p) = \binom{p}{1} u_n(1) + \binom{p}{2} u_n(2) + \ldots + \binom{p}{p-1} u_n(p-1) + u_n(p),$$

since, for each value of r, there are $\binom{p}{r}$ ways of choosing r colours and so $\binom{p}{r} u_n(r)$ ways of painting the disc with exactly r colours. Considering the coefficient of x^{p-r} in the product of $(1+x)^p$ and $(1+x)^{-(r+1)}$ we find, for $r>p$,

$$\binom{p}{r} - \binom{p}{1}\binom{p-1}{r} + \binom{p}{2}\binom{p-2}{r} - \ldots + (-1)^{p-r}\binom{p}{p-r} = 0,$$

and therefore

$$u_n(p) = v_n(p) - \binom{p}{1} v_n(p-1) + \binom{p}{2} v_n(p-2) - \ldots + (-1)^{p-1}\binom{p}{p-1} v_n(1).$$

It remains to evaluate $v_n(p)$.

There are $v_{n-1}(p)$ ways of painting the first $n-1$ sectors so that the first and last have different colours, leaving a choice of $(p-2)$ colours for the nth sector, and $v_{n-2}(p)$ ways of painting the first $n-1$ sectors so that the first and last have the same colour, leaving a choice of $(p-1)$ colours for the nth sector, from which it follows that

$$v_n(p) = (p-2)v_{n-1}(p) + (p-1)v_{n-2}(p).$$

Since $v_2(p) = p(p-1)$ and $v_3(p) = p(p-1)(p-2)$ the solution of the difference equation is

$$v_n(p) = (p-1)^n + (-1)^n(p-1).$$

(This answers counts as different ways of painting, all aspects of the disc obtained by rotating it through any number of sectors from 1 to $n-1$.)

R.A.F., Debden. G. HOFFMANN DE VISME

1899. A theoretical study of the potential distribution of a protuberance consisting of dumbbell-shaped spheres indicates that the field will be greatest near the top and outer parts of the spheres.—J. A. Becker, *The Life History of Adsorbed Atoms, Ions and Molecules, Annals of N.Y. Academy of Sciences*, September, 1954. [Per Mr. J. M. Wales.]

2736. Coloured cubes : a new " Tantalizer ".

A puzzle sometimes encountered has the following (or an equivalent) form. Different faces of each of four cubes are coloured each with some one or other of four different colours, all faces being coloured on all four cubes. The four cubes have to be arranged in line, in a square prism $4 \times 1 \times 1$, so that all four colours appear on every 4×1 boundary face of the prism.

The relevant theory has been given by F. de Carteblanche* who points out that it applies equally well to n cubes and n colours, with similar boundary conditions. Separate consideration is given to each of the two pairs of opposite multicoloured boundary faces, in an assumed solution. Cubes which do not contribute the same colour to both faces in a pair must evidently fall into one or more sets, such that when the cubes in each set are taken in a particular cyclic order, each colour concerned will successively appear in opposite boundary faces, once from each of two serially adjacent cubes. The two different pairs of opposite boundary faces will in general require different sets or orderings of the cubes.

The colours are therefore represented by appropriately labelled nodal points of a network, which is then completed as follows. Each individual cube is given a reference numeral, and three lines marked with this numeral are drawn to indicate associated colours for pairs of opposite faces on the cube concerned. When opposite colours are distinct, the line joins distinct nodes of the network ; when opposite colours are identical, a loop is drawn beginning and ending at a single node. When these lines are drawn for all the cubes, the network contains all the information required to identify a cube associated with any particular numeral. (Mirror images, when different, are not distinguished ; but this is immaterial, as nothing prohibits an exchange of colours for the two inessential faces.)

The problem is then reduced to finding two mutually exclusive sets of lines of the network, such that in each set there is a single line for each numeral, without repetitions, while the vertices associated with the ends of the lines consist of the whole set of vertices taken twice over. The whole collection of lines in each set therefore forms either a single closed circuit, or possibly several closed circuits (a loop for opposite faces of like colour being considered a closed circuit). If two such sets can be found, the simplest method of achieving the correct assembly is then perhaps as follows.

We first consider the *remaining* network lines (which in general will *not* form a set of closed circuits). These indicate a method of aligning the cubes, conveniently in numeral order, by axes joining the pairs of coloured faces which will *not* appear in the multicoloured faces of the prism assembly.

We then consider either of the mutually exclusive sets of network lines which form the circuits last discussed. Any closed circuit gives an alternation of colours (nodes) and cubes (numerals), such that the cubes can be taken in the order in which their numerals appear round the circuit, and can as required be rotated about the *long* axis of the assembly, so that finally each later cube has a face of similar colour to one face of its immediate predecessor, with these two faces located in opposite multicoloured faces of the assembly. At the end of a circuit, the first cube fulfils the conditions for a successor of the last, and the arrangement is cyclic. This is done for all circuits in the set, for the same pair of opposite multicoloured boundary faces, if there are multiple circuits.

We now consider the second of the mutually exclusive sets in similar fashion, and satisfy a corresponding condition for the remaining pair of opposite multicoloured faces of the assembly, by taking colours and cubes in the order

* De Carteblanche, F., " The Coloured Cubes Problem ", *Eureka* No. 9 (April 1947), 9.

indicated in each circuit concerned, and giving 180° rotations to the cubes, as required, about axes joining the faces whose colours were determined from consideration of the first set. Here, too, cyclic closure is guaranteed, and at the end the puzzle is solved.

If no such pair of mutually exclusive sets of lines can be found in the diagram for the given cubes, the problem is impossible to solve. If there is a unique pair of such sets, there is a solution which is essentially unique apart from trivial distinctions.

De Carteblanche considers a particular case, and shows it has a unique solution, with a chance 1/41,472 of appearing on random assembly. The current commercial version (the "Tantalizer") is an essentially different case, but similar statements here apply. For both cases, each of the two sets of diagram lines gives a single closed circuit. In neither case, however, are there six faces of each colour, over the set as a whole. De Carteblanche splits the colours 7, 7, 6, 4, and the "Tantalizer" 7, 6, 6, 5. In both cases there are two colours which never appear as opposites, on any cube, and two colours neither of which ever colours a pair of opposite cube faces.

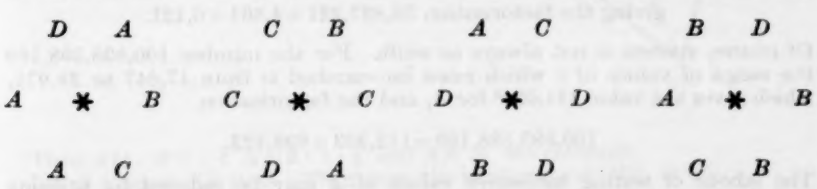
The following set of cubes has rather more symmetry and variety; colours are divided 3, 1, 1, 1 on every cube, and all possible colourings of opposite faces can be found. The colouring rules are simple, both to state, and to follow in manufacture:

(1) Colour two opposite faces and one other face all alike, for each of the four cubes, using a different colour for each cube.

(2) On each cube, take the uncoloured face whose opposite face is already coloured, and colour it differently from the three faces previously coloured, taking care that the *same* colour is used here for *three* of the four cubes; this colour is thus that already appearing on the *fourth* cube, which here receives one of the other colours.

(3) The two opposite faces still uncoloured on each cube then receive the two remaining colours as appropriate in each case.

The network diagram here again allows only one choice of mutually exclusive sets of lines. The chance of success by random assembly is however improved to 1/10,368 (this may be an advantage!) and there are other interesting differences from the other forms. The figure here given will sufficiently



indicate the colouring of the cubes and their manner of assembly in the solution. The solution has features enabling it to be readily memorised.

My thanks are due to de Carteblanche, for the theory which I have given above and for the recommendation to explore these questions further. (I would be intrigued to know his or her relation, if any, to *Blanche Descartes*!)

THOS. H. O'BEIRNE

2737. Factorising large numbers and the verification of primes.

Numbers of the form $6N+1$ have either a pair of factors of the form $6n+1$ or a pair of the form $6n-1$.

If $6N+1=(6a+1)(6b+1)$ and $N=6q+r$, $0\leqslant r<6$, then $6q+r=6ab+(a+b)$ so that $a+b-r$ is divisible by 6.

Let $a+b-r=6z$, then $ab=q-z$ and a, b are the roots of the quadratic equation

$$t^2-(6z+r)t+(q-z)=0.$$

For rational roots we require that $\Delta_1=(6z+r)^2-4(q-z)$ be a positive perfect square. For large q, to make Δ_1 positive for positive z we must take $z\geqslant\frac{1}{8}\sqrt{q}$; moreover $z\leqslant q$ which greatly reduces the range of values of z to be considered. If $6N+1$ has no prime factor less than p, the maximum value of z may be shown to be $q/p+p/36$.

If $6N+1=(6a-1)(6b-1)$ the values of a, b are similarly the roots of the quadratic equation

$$t^2-(6z-r)t+(q+z)=0,$$

and the condition for rational roots is now that

$$\Delta_2=(6z-r)^2-4(q+z)$$

be a positive perfect square, and for Δ_2 to be positive we must take $z>\frac{1}{8}\sqrt{q}+1$. Since $\Delta_2<N^2$ it follows also that $z\leqslant q+1$. If $6N+1$ has no prime factor less than p, the maximum value of z is again $q/p+p/36$, for $p>7$.

Example. To factorise $30,827,221=6\times5,137,870+1$
$=6(6\times856311+4)+1$.

The corresponding values of z and Δ, beginning with $z=308$, the smallest value of z for which $9z^2\geqslant856311$, are

z	308	309	310	311
Δ_1	5,892	28,156	50,492	72,900
Δ_2			18,252	

Since $72,900=270^2$ the search is at an end and we find $a=800$, $b=1,070$

giving the factorisation $30,827,221=4,801\times6,421$.

Of course, success is not always as swift. For the number 100,895,598,169 the range of values of z which must be searched is from 17,647 to 28,075, which gives the value $131,020^2$ for Δ_1 and the factorisation

$$100,895,598,169=112,303\times898,423.$$

The labour of testing successive values of Δ may be reduced by utilising periodicities in the successive differences, bye-passing ranges of values with terminal digits 2, 3, 7, 8 which cannot be squares, and also those in which the remainders of the divisions by 9 or 11 or even 7 or 13 show that they cannot produce perfect squares.

A similar method is, of course, applicable to numbers of the form $6N-1$, with factors $6a+1$, $6b-1$; for $a<b$ the corresponding value of Δ_1 is

$$(6z+r)^2+4(q-z),$$

and since this is positive for $z\geqslant0$, the range of values of z to be considered starts from $z=0$. Since $\Delta_1\leqslant N^2$ it may be shown that $z\leqslant q$, and if no prime factor is less than p then $z\leqslant q/p-p/36$. For the number 69,177,893, for instance, Δ_1 is a perfect square for $z=64$, giving the factorisation $7,243\times9,551$.

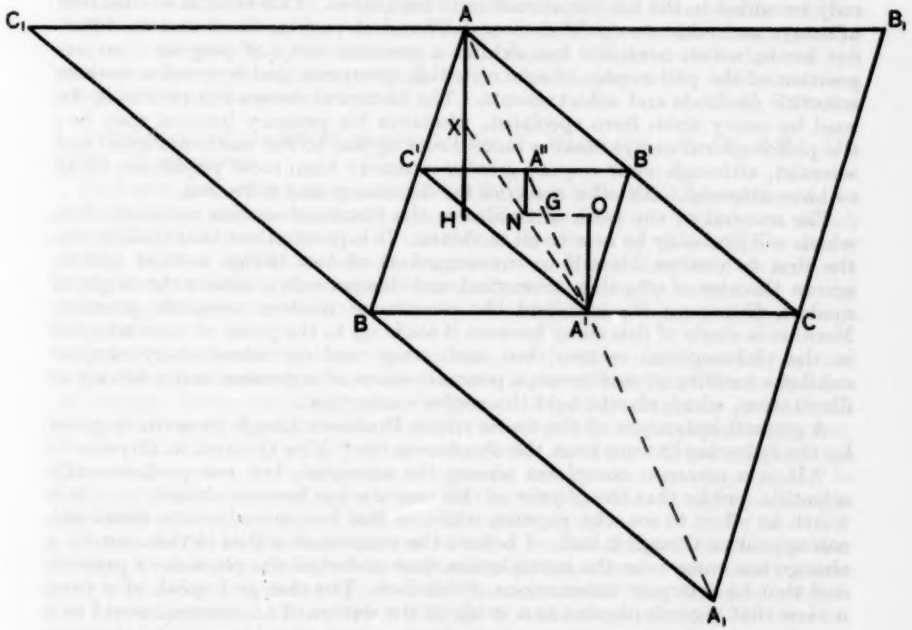
Rio de Janeiro, Brazil W. L. ALDRIDGE

2738. The nine-point Circle.

A method of teaching nine-point circle work which gives easily remembered proofs is to consider first a $\triangle ABC$, together with the $\triangle A'B'C'$ formed by joining its mid-points of its sides, and the $\triangle A_1B_1C_1$ formed by drawing lines through its vertices parallel to its sides. Then the $\triangle$s $A_1B_1C_1$, ABC, $A'B'C'$ are similar, and corresponding lengths are in the ratios $2 : 1 : \frac{1}{2}$.

Now the circumcentre of $\triangle A_1B_1C_1$ is the orthocentre H of $\triangle ABC$.

Let O be the circumcentre of $\triangle ABC$, and N the circumcentre of $\triangle A'B'C'$. Let A'' be the mid-point of $B'C'$.



Then $AH : A'O : A''N = 2 : 1 : \frac{1}{2}$, and $AA''A'$ are collinear.

Let AA' meet NO in G. Then from the similar $\triangle$s $A''NG$, $A'OG$,

$$A'G = 2GA'',$$

whence $A'G = \frac{1}{3}AA'$, and G is the centroid of $\triangle ABC$ (and also of $\triangle$s $A_1B_1C_1$, $A'B'C'$).

Let $A'N$ meet AH in X. From the $\triangle$s $A'AX$, $A'A''N$, $AX = 2A''N = \frac{1}{2}AH$. Also $XN = NA'$. So the circumcircle of $\triangle A'B'C'$ passes through X, the mid-point of AH, and, since $A'X$ is a diameter, also through the foot of the perpendicular from A to BC.

Throughout the teaching, once the pupil has grasped the "Chinese box" relationship of the three $\triangle$s $A_1B_1C_1$, ABC, $A'B'C'$, the main emphasis should be on the three lengths AH, $A'O$, $A''N$. With these "guiding lines" the proof of any required nine-point circle property can be fairly easily reconstructed.

The Observatories, Cambridge. JOYCE LINFOOT

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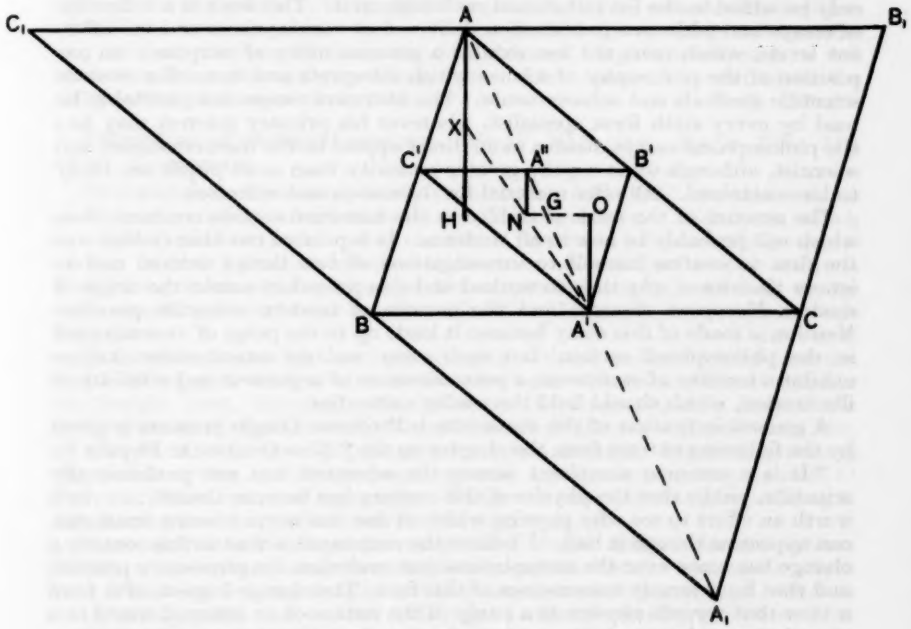

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REVIEWS

The Scientific Adventure. By HERBERT DINGLE. Pp. 372. 30s. 1952. (Pitman)

In an epilogue to the *Report on the Teaching of Higher Geometry in Schools* (1953), attention was called to the importance of broadening the education of specialists in the sixth form by general discussions of the relation between science and abstract mathematics and to suggestions of some of the relevant available literature enumerated in the Association's *List of Books suitable for School Libraries* (1954). Professor Dingle's *Scientific Adventure* should not only be added to the list but should rank high on it. This book is a collection of essays and addresses published or delivered at various times and on different levels, which none the less exhibit a genuine unity of purpose: an exposition of the philosophy of science which interprets and reconciles modern scientific methods and achievements. The historical essays can profitably be read by every sixth form specialist, whatever his primary interest may be; the philosophical essays make a more direct appeal to the mathematician and scientist, although some require greater maturity than most pupils are likely to have attained. All offer material for discussion and reflection.

The account of the work of Galileo in the historical section contains ideas which will probably be new to all students. It is pointed out that Galileo was the first to confine himself to investigations of *how* things worked and to ignore theories of *why* they so worked and this procedure marks the origin of modern European thought and the essence of modern scientific practice. Mention is made of this essay because it leads up to the point of view adopted in the philosophical section, but each essay and an introductory chapter exhibit a lucidity of statement, a persuasiveness of argument and a felicity of illustration, which should hold the reader's attention.

A general indication of the thesis which Professor Dingle presents is given by the following extract from the chapter on the "New Outlook in Physics":

"It is a common complaint among the educated, but not professionally scientific, public that the physics of this century has become absurd.... It is worth an effort to see why physics, which in fact has never become irrational, can appear as though it had. I believe the root cause is that in this century a change has come over the metaphysics that underlies the physicist's practice and that he is largely unconscious of this fact. The change I speak of is from a view that regards physics as a study of the nature of an external world to a view that regards it as an attempt to find rational relations between the elements of our experience. The older physicist believed in Nature and thought of himself as making experiments to see what She was like. She was there whether he could observe her or not. But the modern physicist thinks first of all of what he observes in his experiments and is not interested in anything that he cannot possibly observe. He looks for relations between his observations and ignores everything else. But he still expresses his results as though they were discoveries of the essence of Nature, because he is so used to this way of speaking that he does not realise that his discoveries no longer conform to it. When they are expressed as the characteristics of a world existing outside us and independently of us, which causes our experience by its impact on our sense organs, these discoveries require such a world to have contradictory properties. Hence, by retaining this form of expression, the physicist finds himself presenting his perfectly rational achievements as though they were nonsensical."

A scientific law is a means of correlating experiences, and the pattern to which it refers is built round *concepts*. As professor Coulson has said recently: there is no force of gravitation except in our own minds as we try to compre-

hend the falling stone ; there is no electron except in our imagination as we seek to understand the behaviour of a wireless valve.

Professor Dingle's ideas were developed in a course of lectures which he gave more than 20 years ago in America and were published under the title, *Through Science to Philosophy* ; they are re-stated with additional material in his *Scientific Adventure*. It is hard to understand why they have not attracted more attention, either by way of support or alternatively by serious critical analysis. It may be that their acceptance requires too radical a change in general outlook ; on the one hand, most philosophers have only a superficial acquaintance with mathematics and science, although of course there are notable exceptions ; on the other hand, most scientists are too preoccupied with producing results to feel inclined to stand aside and examine the implication of the work they have done. The purpose of this review is not to make any critical estimate of the value of Professor Dingle's work, but to call attention to a thought-provoking book which cannot but stimulate the imagination of those who read it. C. V. DURELL

Foundations of analytical geometry. By W. M. SMART. Pp. ix, 249. 14s. 1956. (Longmans)

Professor Smart has temporarily deserted his usual field of astronomy to give us an excellent introduction to analytical geometry. This book, which covers the syllabus for A level of the G.C.E., has been written with an appreciation of the difficulties which beginners sometimes find in this subject. Everything is clearly explained, and the pace is leisurely, especially in the early chapters, so that readers will not feel that they are being rushed through the course.

The author starts right at the beginning with an explanation of how the coordinate system is set up and the idea of a locus. Then follow three chapters on straight lines, including pairs of lines. A pleasing feature here is the interpretation of the sign in the formula for the distance of a point from a line, an idea which even good pupils sometimes find difficult. Three chapters are devoted to circles, including coaxal systems, and finally one chapter each to the parabola, ellipse and hyperbola. Numerous examples are given throughout, some solved and others left for the reader to solve.

There are three small points on which I would differ from the author. Firstly, some rather unimportant topics are included in the text which might have been relegated to examples. For instance, a whole section is devoted to the general equation of a circle touching an axis of coordinates. A student working by himself might give such topics more weight than they are worth.

Secondly, in discussing the equation of any straight line through the point of intersection of two given straight lines, the author gives, as his first method, one which depends on finding the point of intersection of the given lines. Now admittedly this is harmless enough in dealing with two lines, but one would not wish pupils to use this method later on in finding, for example, the equation of any circle through the points of intersection of a given circle and straight line. It would perhaps have been better not to mention this method at all.

Thirdly, Professor Smart defines a conic by the focus-directrix property, but when he comes to the ellipse, he *states* the standard form $x^2/a^2 + y^2/b^2 = 1$, shows that it can be derived from the equation of a circle by reducing the ordinates of the circle in the ratio b/a, and only much later does he prove that the standard form follows from the focus-directrix property. Why not do this at the beginning?

The book is well printed, with clear diagrams and the most important formulae in heavy type. It can be strongly recommended either as a complete course in itself or as a prelude to more advanced work. E. J. F. PRIMROSE

An introduction to analytical conics. By C. T. RAJAGOPAL and V. R. SRINIVASARAGHAVAN. Pp. xxi, 320. 9s. 6d. 1955. (Geoffrey Cumberlege, Oxford University Press)

This book, which is intended for Indian undergraduates, follows a somewhat different procedure from most text-books on conics. Starting with a rather unusual definition of a conic, the authors derive the three standard forms, and show that the general equation of the second degree can be reduced to one of these forms by a suitable transformation. General theorems on conics are then proved, using the usual abridged notation (S, S_1, S_{12}, and so on), and finally those results which cannot easily be derived from the general equation are proved by using the standard forms. This procedure is certainly shorter than that of treating the three types of conic in detail separately, where many theorems are proved three times over.

Nevertheless, the book is long, and there are two main reasons for this. Firstly, the authors pursue the theory in great detail, proving a large number of subsidiary results. Secondly, there is a great deal of work with oblique coordinates: the small gain which results from their use hardly justifies the time spent.

My chief criticism of the book is that there is a certain lack of precision in some places, particularly in definitions. For instance, the symbol (P) for a point is not properly defined: and it seems to be both homogeneous and non homogeneous at the same time. Again, points at infinity are discussed by introducing a symbol ∞, which is to satisfy certain conditions. A much more satisfactory method is to introduce homogeneous coordinates.

A pleasing feature is the frequent reference to the history of the subject. This book may prove to be very useful in Indian universities.

E. J. F. PRIMROSE

Tafeln zum Vergleich Zweier Stichproben mittels X-test und Zeichentest. (Tables for Comparing two Samples by X-test and sign test). By B. L. VAN DER WAERDEN and E. NIEVERGELT. Pp. 34. DM 4.80. 1956.

The present work contains tables which are intended to deal with situations like the following, given by the authors as an example: In an industrial organisation the following widely scattered waiting times are measured:

11, 34, 13, 18.

After reorganisation the waiting times appear to be shorter, and less scattered:

8, 10, 7, 6.

We wish to judge whether there has been a genuine reduction of waiting times. The applicability of Students t-test is doubtful, because of apparent non-normality and inequality of variances.

There is an introduction in German, followed by six tables and a section in English on the use of the tables. Table 1 is trivial. Table 2 gives $\psi\left(\frac{r}{n+1}\right)$ for $r=1(1)n$, $n=6(1)50$, to two decimal places, where $z=\psi(u)$ is defined by

$$u=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-t^2/2}\,dt.$$

Table 4 is a two-place table of $\psi(x)$ for $x=0(\cdot001)\cdot99$
Table 5 is a three-place table of

$$Q=\frac{1}{n}\sum_{1}^{n}\psi^2\left(\frac{r}{n+1}\right)$$

for $n = 1(1)\ 150$. Table 6 is a table of $k(n\ ;\ \beta)$, where k is defined as the least integer for which the binomial sum

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{k}$$

is greater than $\beta.2^n$, for $\beta = \cdot025,\ \cdot01,\ \cdot005$, and $n = 5(1)100$. Table 3 gives three percentage points of the X-test for two samples of combined size $n = 5(1)100$.

The experienced statistician confronted with data such as those quoted above would consider carefully the nature of the waiting times observed, and the changes involved in the reorganisation, with a view to judging whether some transformation should be applied to the figures. For example, it might appear that the effect of the reorganisation might be expected to consist mainly in an increase in the effective number of "servers" in the system, which would result in dividing the original waiting times x by a constant a, giving the new waiting time y as

$$y = \frac{x}{a}.$$

If this were so, a logarithmic transformation would be appropriate, since

$$\log y = \log x - \log a$$

and we would apply a t-test to the logarithms :

$$\xi = \log x :\ 1{\cdot}04,\ 1{\cdot}53,\ 1{\cdot}11,\ 1{\cdot}26$$
$$\eta = \log y :\ 0{\cdot}90,\ 1{\cdot}00,\ 1{\cdot}00,\ 0{\cdot}85,\ 0{\cdot}78$$

and (with the usual symbolism) we would find $d = \xi - \eta = \cdot3525$ with an estimated standard error of $\cdot1179$, on 6 degrees of freedom. The value of t corresponding to 5 percent probability is 2·447, so that the 5 percent fiducial limits for d are

$$\cdot3525 \pm 2{\cdot}447 \times 1179, \text{ or } \cdot064 \text{ and } \cdot641,$$

corresponding to an effective increase of servers in a ratio a between 1·16 and 4·38. In particular, the hypothesis of no change can be included, since it corresponds to a value of t of 3·00, or a probability of about ·03.

But it might be that the main change expected as a result of reorganisation would not be expressible as a proportional reduction of waiting times, or anything so simple, and we might be in the position where our scale of measurement bore no simple relationship to the quantities being observed, merely serving to rank them in order of magnitude. In this case, following Fisher and Yates, we would use their Table XX and replace the ordered values

$$6,\ 7,\ 8,\ 10,\ 11,\ 13,\ 18,\ 34$$

by the mean values of the 1st, 2nd, etc. observations in a random sample of 8 observations from a standard normal population :

$$-1{\cdot}42,\ -0{\cdot}85,\ -0{\cdot}47,\ -0{\cdot}15,\ +0{\cdot}15,\ +0{\cdot}47,\ +0{\cdot}85,\ +1{\cdot}42$$

and then carry out our analysis on these new figures. Because of lack of normality we would not expect the probabilities found from the t-test to be exact, but the errors would not be very serious. In the present case we find $t = 3{\cdot}74$, giving a probability of rather less than 0·01, somewhat exaggerating the significance of our data. A test such as this, depending only on the ranking of the observations, is called a rank order test.

Van der Waerden proposes another rank order test, the X test, in which the rth observation out of n is replaced, not by the mean value of the rth observa-

tion from a standard normal distribution, but by the standard normal deviate corresponding to a probability of $r/(n+1)$. Since the latter is quite a good first approximation to the former, in samples of moderate size, van der Waerden's test will be almost identical with that of Fisher and Yates, and it is difficult to see why the X-text should be put forward as new. Fisher and Yates' test is not mentioned in van der Waerden's introduction, where the X-test is compared with the simple t-test and with Wilcoxon's rank order test; and no reference is made to the work of Wassily Hoeffding on the selection from among the infinitely many possible rank order tests. Hoeffding's work shows Fisher's test to be, in a sense, best possible. One is led to conclude that, although the authors of the present tables refer to Fisher and Yates Table X, the latter's Table XX must have escaped their notice.

In so far as Table 3 gives exact percentage points for the X-test for small sample sizes, there is a slight advantage over the Fisher-Yates procedure, which is liable to be slightly in error because of non-normality. However, since rank order tests should only be used as a last resort, when the more fruitful analysis in terms of t, with or without functional transformation, cannot be made, and since the existing Fisher-Yates tables provide a rank order test at least as good as the X-test, Tables 1–5 cannot be said to represent a necessary part of the statistician's library.

The binomial table 6 is intended to facilitate a test of any hypothesis which asserts that two possible outcomes are equally probable. The authors remark that such a test was effectively used by Dr. Arbuthnot in 1710, and it has been used many times since then. Here again the reviewer is unable to recall a case where he wanted to use this test and found the normal approximation inadequate; and the use of the normal approximation has the advantage of enabling the actual level of significance attained to be estimated. This would guard against the fallacy involved in regarding as significant a result giving a probability of ·049, while rejecting as insignificant one giving a probability of ·051. Such senseless, mechanistic thinking, deplorable though it is, is to be found in some quarters, and the presentation of tables in this form runs the risk of encouraging it.

It must be said, however, that, with all their faults, the publication of these tables in Germany is to be welcomed as further evidence of the spread of statistical methods into that largely virgin soil.

Equation (3) on p. 4 contains an obvious misprint. G. A. BARNARD

Methods in Numerical analysis. By K. L. NIELSEN. Pp. xiii, 382. 48s. 6d. 1956. (The Macmillan Company, New York and London)

Before the war, numerical mathematics was a loose body of material of interest principally to astronomers, geodetic surveyors, actuaries and a few mathematicians of eccentric tastes. Apart from the German text by Runge and König and *The Calculus of Observations* by Whittaker and Robinson, there was not much literature since Gauss.

Introductions to the whole modern field became available in the *Numerical Calculus* of Milne (1949) and the *Numerical Analysis* of Hartree (1952); both these authors had first-hand experience of war-time developments including linear algebra and numerical differential equations. Since then the previous dearth of literature has been replaced by something of a surfeit, with about one new book per month, as the potentialities of large-scale calculators became appreciated, and with these the importance of numerical analysis in technological education.

The work here reviewed is intended as an elementary textbook "for the practical man" in the numerical field, avoiding "mathematical sophistication" and giving detailed accounts of systematic procedures, with illustrative examples. The best parts of the book are perhaps in Chapter VI (principally

the "Crout" method for sets of linear equations, also successive approximation for one or more non-linear equations) and in Chapter VII (an introduction to numerical solution of differential equations). Much of this is computing technique described with economy of theoretical justification.

Elsewhere the finite-difference material is made to seem more tedious than need be: symbolic operators are barely mentioned; the author seems averse to central-difference notation, and uses a non-standard form when surrender becomes inevitable. A suffix has a negative sign omitted in an important quadrature formula (42.2). The Graeffe root-squaring procedure is described with no mention of negative or complex roots. Least-squares fitting seems rather unnecessarily complicated, even with orthogonal polynomials (tabulated for up to 33 ordinates, from Fisher and Yates), let alone by the tables of the Nielsen-Goldstein method.

An organisation with ample library funds might acquire this book; other works would have a greater claim when resources are limited. A student (whether of technology or of mathematics) will obtain considerably more, for three-quarters of the price, from Hartree or Milne. These meet the needs of the practical man at least as well (if not better), with no greater demands on his previous knowledge, in comparable contexts. THOS. H. O'BEIRNE

Champs de Vecteurs et de Tenseurs: introduction a l'électromagnetisme. By EDMOND BAUER, Pp. vii, 204. 1955. (Masson & Cie, Paris)

This book, written "as an introduction for the young physicist or as a complement for the young mathematician", is an excellent example of a lucid presentation of mathematics for the physicist. For the proper understanding of physics it is essential that one should first acquire a precise knowledge of the necessary mathematical language, and nowhere, perhaps, is this more so than in dealing with Electromagnetism and Relativity; here the basic language is that of vectors and tensors and the book under review provides one of the clearest and most concise accounts of these topics. The book is divided into five chapters, the last of which is concerned with the application of vector and tensor field-theory to electromagnetism. The first chapter deals with the algebra of cartesian vectors and tensors; in it the author has been at pains to stress the essential distinction to be drawn between axial and polar vectors, a fact which is sometimes overlooked in the literature. It also contains several useful illustrations drawn from geometry and mechanics. The second chapter is devoted to vector and tensor fields; in this the differential operators are introduced and their significance clearly stated. The next chapter is concerned with what Professor Bauer terms Newtonian and Laplacian fields, a Newtonian field being one in which the curl of the intensity of the field, but not the divergence, is everywhere zero, and a Laplacian field, one in which the divergence, but not the curl, of the intensity vanishes everywhere. The distinction and essential difference between these fields is carefully discussed and the fallacy, often stated, that in a Laplacian field the "lines of force" of the field are closed, is avoided. In the fourth chapter the theory of vector and tensor fields is extended to a generalised space and the reader will find this an excellent introduction to the more advanced treatises on this subject.

Throughout this book, Professor Bauer has been highly successful in presenting new ideas clearly and incisively. The argument is everywhere sound and rigorous and though one might not favour the method of derivation of the force exerted by a magnetic field on a linear current on Page 151, elegant though this is, the treatment of the troublesome question of the energy in a magnetic field is admirable. The book is well produced, well printed and is very readable; it is recommended to all students as an excellent and concise account of the subject and one which they should not fail to read.

V. C. A. FERRARO

Report of a conference on mathematical education in South Asia held in Bombay in February 1956.

This conference was the first of its kind to be held in Asia and was attended by about 75 mathematicians from twenty countries, who discussed the problems of mathematical education at all levels. The United States representative Marshall Stone spoke of the need to train mathematical technologists and the difficulties in a common course for mathematicians and technologists. Choquet stressed the message which modern mathematics and research have for the teacher and urged every teacher to "kill the old man in himself and find out that he can be a discoverer and that his pupils also can be discoverers, and in fact much more powerful than himself". Broadbent spoke both on English Education and on mathematical typography. He stressed the influence that the Universities in this country exercise on the school curriculum through the General Certificate of Education. With characteristic wit he traced the influence of the Mathematical Association on the development of geometrical teaching and the chaos which first followed the rejection of Euclid as a text. The case for the fundamental place which mathematics should hold in the Educational system is based, he contended, not on mere utility, nor even on the role which mathematics plays in modern technology, but on the fact that mathematics is an integral part of modern culture.

Russia and Italy presented to the Conference detailed schedules and time tables, including the number of hours spent on the several branches of mathematics in lectures and exercises. Oppenheim, speaking for Singapore and the Malaya Federation, made life in an English University or School seem like paradise by comparison, and stressed the important part played by the Malayan Mathematical Society (affiliated to the Mathematical Association) and its mimeographed bulletin.

R. L. GOODSTEIN

German-English mathematical vocabulary. By SHEIL MACINTYRE and EDITH WHITE. Pp. 95. 8s. 6d. 1956. (Oliver and Boyd)

The little book contains a 50 page vocabulary and 40 page grammatical sketch, a list of common abbreviations and alphabets in Gothic and German script; the grammatical sketch is the work of Lilias Brebner and its inclusion makes the vocabulary available for complete novices in the German Language. For its size the vocabulary is remarkably comprehensive and tested on an encyclopaedia of elementary pure mathematics scored about 90% of successes. The preface explains that applied mathematics, statistics and mathematical logic are not included. Nearly all compound words are omitted and must be looked up by their component parts; one or two peculiar exceptions to this rule were noted, e.g. *Freiheitsgrad* is included, and so is *grad*, but not *Freiheit*.

R. L. GOODSTEIN

Ordinal Algebras. By A. TARSKI. Pp. 133. 27s. 1956. (North Holland Publishing Company, Amsterdam)

The main purpose of this book is described as being the development of the theory of ordinal addition for arbitrary relation types. The method used is that of abstract algebra. Instead of studying the ordinal sums of relation types directly the first concern is with a system which comprises a set of arbitrary elements, which includes the element 0, an operation Σ on finite and infinite sequences of these elements, an operation $+$ on couples of elements and an operation $*$ on single elements. Ordinal algebras are a class of such algebraic systems satisfying certain postulates of an arithmetical character. These postulates are satisfied by taking the set of all reflexive relation types for elements, the familiar operations of ordinal addition for Σ and $+$, the type of the empty relation for 0 and the operation of conversion (forming the converse) for $*$.

The last 50 pages of the book comprise two appendices. The first by Chen-Chung Chang contains some additional theorems on ordinal algebras and the solution of a problem left unsolved in the text. The second, by Bjarni Jonsson is on a unique decomposition theorem for relational addition.

R. L. GOODSTEIN

Complete Theories. By ABRAHAM ROBINSON. Pp. 129. 27s. 1956. (North Holland Publishing Co., Amsterdam)

This book continues the author's work on the integration of metamathematical results and modern algebra, described in his 1951 tract "On the metamathematics of algebra". The fundamental new concept of the present book is *model completeness*. A non-empty set K of statements of the lower predicate calculus is said to be model complete if for every model (mathematical structure) M which contains no relations other than those in K and for every statement X defined in M, either X holds in all extensions of M which are models of K or the negation of X holds in all such extensions. The concept of model completeness is not comparable with the classical concept of completeness, since for instance a set of axioms for an algebraically closed field may be given which is model complete, but not complete, and the converse that there are complete sets which are not model complete is almost trivial. Amongst theories which are proved to be model complete are the elementary theory of completely divisible torsion-free abelian groups (of at least 2 elements), ordered abelian groups which are completely divisible, the elementary theory of real closed fields, and the elementary theory of an algebraically closed field with non-trivial valuation in an ordered abelian group.

R. L. GOODSTEIN

Einfuhrung in die hohere Mathematik. By K. STRUBECKER. Vol. I. Grundlagen. Pp. xv, 821. DM 36. 1956. (Oldenburg, Munich)

This volume contains four parts, on number and number theory; elementary algebraic functions; limits and infinite series; elementary transcendental functions, continuity and inverse functions. It is a rich storehouse of information, full and detailed without being indigestible. Genial is perhaps the word which best describes its style. Explanations are careful, complete and reliable; every topic is illuminated by historical references. The diagrams, of which there are more than 300, are magnificent.

It would take up too much space to list even a small fraction of the contents, but especially noteworthy are the sections on the theory of errors, the theory of groups, the solution of algebraic equations, the projective plane, and the theory of cycloids. To write a book of this size and scope is a tremendous undertaking and to keep it interesting and readable as Strubecker has done merits high praise. The only omission which seems fair to note is the lack of any account of convex functions; on a point of notation the use of $GO(10^{-n})$ to denote a decimal zero to n places of decimals, seems rather clumsy and less expressive than the reviewer's $0(n)$.

R. L. GOODSTEIN

Differential Equations. By S. V. FAGG. Pp. 128. 7s. 6d. 1956. (English Universities Press)

This little book is designed to interest the Sixth Form Science specialist reading for G.C.E. Advanced level and should be very successful in achieving this aim. The treatment is simple, practical and straightforward, with numerous illustrations from mechanics, electric circuit theory and engineering. Explanation is full and helpful and drill exercises plentiful. It is surprising in so elementary a text to find a chapter on the Laplace Transform, but the innovation reflects an important modern trend in the treatment of differential equations. The book may be warmly recommended as a first school text.

R. L. GOODSTEIN

Differential Equations. By H. W. REDDICK and D. E. KIBBEY. 3rd Ed. Pp. 304. 36s. 1956. (Wiley, New York; Chapman Hall, London)

This is a very readable introduction to the standard elementary methods of solution of ordinary differential equations, with a brief reference to partial differential equations. Existence questions are not discussed. Examples are ample and interesting with applications to mechanics, electromagnetic theory and engineering.

The section on linear constant coefficient equations could be amplified with advantage to include some reference to operational methods. The account of the solution in series of the equation $x^2y'' + pxy' + qy = 0$ considers only the case when the roots of the index equation do not differ by an integer.

R. L. GOODSTEIN

La Cinématique Relativiste. By H. ARZELIÈS. Pp. xi, 228. 2,500 fr. 1955. (Gauthier-Villars, Paris)

According to his preface, Professor Arzeliès is writing the work of which this is the first volume in order to remedy the lack of a French textbook of relativity theory for university students and their teachers. There is not the same lack for English-speaking students. Also what seem to be the special merits of the book are ones that actually might not make it ideally suited as a straightforward introduction to a working knowledge of the subject. For Professor Arzeliès "writes round" the subject a good deal and also provides very careful discussions of some of its possible difficulties and alleged paradoxes. Moreover, at the end of each chapter he supplies full and illuminating historical notes and extensive bibliographies. At any rate for English readers, these discussions and notes will probably prove to be the chief value of the book. For, naturally, people still continue to think up difficulties and puzzles in the subject. But at the present stage in its history almost everything of this sort has been thought of and written about in the past. Most of what has been thought and written in this way can be found mentioned in this book or can be traced in the references it supplies. I do not know of any other book of which this can be said.

The present volume deals only with special relativity and only with the part of the theory that can be included under the heading of kinematics. Of course, as Professor Arzeliès is well aware, even in the special theory it is not strictly possible to separate kinematics and mechanics, since the whole point of the theory is to deal only with observations that can be made with real physical apparatus. The distinction is only one that can be made roughly for practical purposes of exposition. While the presentation is highly original, the mathematical methods used are those that have become traditional in the subject. This is in contrast to the book by J. L. Synge (*Relativity : the special theory*, Amsterdam, 1956) recently reviewed in these pages, in which the treatment is mainly by means of the geometry of space-time.

W. H. MCCREA

Atoms and the Universe. By G. O. JONES, J. ROTBLAT and G. J. WHITROW. Pp. 254. 25s. 1956. (Eyre and Spottiswoode)

The rapid advance of modern science in recent years has made it increasingly difficult for the layman to have even a sketchy understanding of the subject. This is particularly true of Nuclear and Relativistic Physics, where the fundamental concepts are so difficult. Many popular expositions are doomed to failure because emphasis is laid on parts of the subject which permit of a spectacular result, regardless of whether it could possibly be understood. The layman today does not want to be amazed; he wants to be reassured that science is at least rational, even though it is becoming more and more complicated.

The authors of *Atoms and the Universe* give the general reader this reassurance in a highly commendable way. Typical of their concern for the reader's difficulties, is that they point out that a distant nebula, the size of our own galaxy, receding at the " fantastic " speed of 40,000 miles per second nevertheless takes about half a million years to pass through a distance equal to its own diameter. The reader is never expected to accept a result without a reasonable indication of the justification for it and at no time does the eminence of the authors obtrude into the text. The concepts of " light horizon " and " curvature of space " are hardly mentioned, without detracting much from the cosmological arguments except that the distinction between the Milne and Eddington interpretations of the expansion of the universe are then not really apparent. The general reader is however, less likely to be troubled by this than by the concepts themselves.

The book is divisible into two sections: Nuclear Physics and Stellar structure and Cosmology. There is a connecting chapter concerning extranuclear structure and the solid state. The brevity of this chapter, compressing as it does, everything between nuclei and stars into about ten pages, may seem a little out of balance unless the title of the book be taken literally.

C. A. HAYWOOD

Errors of Observation and their Treatment. By J. TOPPING. Institute of Physics Monographs for Students.

In just over a hundred octavo pages, the author has presented all the information needed by most experimenters to attach confidence limits to the results of their experiments. To students already familiar with the problems of statistics, the book will prove to be very handy, both because of its size and because of the ease with which formulae connecting the various statistical concepts can be located.

The beginner, however, is less well catered for. The explanation of sampling theory is too short to be adequate and more emphasis could with advantage have been put on the concept of percentage confidence limits.

However, the experimenter cannot fail to find what to do and how to do it when requiring to treat his errors of observation.

The text is freely interspersed with worked examples and exercises and should certainly fulfil the author's hope to appeal " to students of Physics, and perhaps Chemistry, in the sixth forms of Grammar Schools and in the Universities ".

C. A. HAYWOOD

Examples in Mechanical Vibrations. By J. HANNAH and R. C. STEPHENS. Pp. 152. 18s. 1956. (Edward Arnold)

The authors have written a useful little book for students of engineering. The price is not too high these days to pay for a text which covers a small part of their examination syllabus because vibrations will play an important part in their work as engineers.

The book is divided into seven chapters: Free vibrations, transverse vibrations of beams, whirling shafts, torsional vibrations, damped vibrations, forced vibrations and forced-damped vibrations. Each chapter opens with a brief introduction to the theory followed by worked examples which are a reasonable cross-section of a large selection left as an exercise for the student. Some 230 examples are included of which about 66 are solved. Almost all of them are taken from past papers of the Universities of London and Glasgow and the Council of Mechanical Engineers.

It would be unfair to criticise the theory since it is rightly given very little space, the worked examples being the main feature. However, it is perhaps unwise at this level to pay too much attention to the elementary

approach to S.H.M. at the beginning of Chapter 1, where we read, "these formulae give merely numerical relationships . . . without regard to sign". This is put right, as explained in para. 1.7, but it seems to add a little confusion later on. A minor point in the solution on p. 135, is that the expression, "the maximum amplitude occurs when $da/dp=0$", suggests the calculus should be used to obtain the required result. One would like to have seen more use of the energy equation in the early part of the work and a more mature and mathematical approach to the simultaneous equations on p. 29 in particular, but no doubt engineering students will be better pleased with the methods used.

These are academic criticisms and should not detract from a book which will be of great help to engineering students. A. BUCKLEY

Konstruktive Geometrie fur Techniker. By F. HOHENBERG. Pp. 272. 37s. 6d. 1956. (Springer, Vienna)

This book is based on the lectures on practical Geometry given by the author in the Technische Hochschule in Graz to students of Architecture, Civil, Mechanical and Electrical Engineering. It is much more than a collection of rules for obtaining the drawings required by technologists. The author is careful to point out the under-lying theory, even to the extent of digressing into a discussion of the application of the complex variable to Geometry. The author discards the usual somewhat dreary approach to this subject through the medium of numerous exercises in plane Geometry which, to these students, often seem to bear little relation to their other work, and stresses instead, right from the beginning, the necessity of being able to obtain plans, elevations and sections of solid bodies.

The book is divided into three sections. In Section A (126 pages) the reader is introduced to the simple geometrical constructions associated with plans and elevations of straight lines, planes, circles, spheres, sections of cones, pyramids, axometric projection, perspective, reconstruction of plans and elevations from photographs. Numerous examples are given of the applications of these constructions to such varied objects as cranes, bridges, various roof forms, sundials, staircases, various parts of machines etc. The work is extended in Section B (70 pages) to deal with intersections of surfaces, differential geometry of plane and twisted curves and surfaces, conoids, and surfaces generated by moving curves, and figured plans. Here again, all the examples are related to practice, *e.g.* to mountain road construction, pipework, etc. In Section C (50 pages) the book deals with the applications of Geometry to kinematics in two and three dimensions, including work on toothed gearing.

This book, whilst presupposing no previous knowledge of the subject, covers a very large field in some 250 pages and is more in the nature of a treatise than of the routine type of textbook to which one is accustomed. The diagrams are somewhat small, and only exceptionally good reproduction makes them reasonably easy to follow. The printing and the general format make the book a joy to handle. The price makes it attractive to the serious student and to the mathematician—in particular the geometer who is interested in the technical significance and application of his subject.

F. T. CHAFFER

Asymptotic Expansions. By A. ERDÉLYI. Pp. 108. $1.35. 1956. (Dover, New York)

This monograph is a republication of a Technical Report prepared for the U.S. Office of Naval Research. It is based on a course of lectures given by the author at the California Institute of Technology. A comparatively wide

range of topics is treated in a small space, so that the treatment is necessarily somewhat brief in places. Where proofs are omitted they can, however, be readily supplied, or recourse can be made to original papers as the work is fully documented throughout.

After an initial introductory chapter on asymptotic series the author describes various methods for approximating to integrals, such as integration by parts and the methods of steepest descents and stationary phase. In the third chapter the behaviour of solutions of differential equations in the neighbourhood of the point at infinity is discussed. It is shown that the formal solutions, which are in general divergent, are asymptotic solutions. Stokes phenomenon is described and the results are illustrated by considering Bessel functions of order zero. In the last chapter the author considers solutions of ordinary homogeneous linear differential equations of the second order containing a large parameter λ, such as Liouville's equation

$$y'' + \{\lambda^2 p(x) + r(x)\} y = 0.$$

A zero c of $p(x)$ is called a transition point, and the character of the solutions varies according as $x > c$ or $x < c$. On one side of c the solutions are oscillatory while on the other side they are monotonic (for a simple zero). Langer's method of obtaining a single uniform asymptotic representation valid on either side of and at the transition point is described. Applications to Bessel functions of large order and argument are given.

Any mathematician whose work involves the approximation to integrals or solutions of differential equations will find this book invaluable. He will find here a large number of methods carefully described and developed which he would otherwise have to extract, often in a less general form, from several different works.

R. A. RANKIN

The Bequest of the Greeks. By TOBIAS DANTZIG. Pp. 191. 18s. 1955. (George Allen & Unwin)

This is a study of problems, principles and procedures which modern mathematics has inherited from Greek antiquity. It is written for two classes of readers and thus, not surprisingly, is divided into two parts somewhat dissimilar in character.

Part I, " The Stage and the Caste ", is written for those who have " neither the preparation nor the taste for the technical aspects of mathematics but have come to realise its importance for contemporary thought and life ". It may be said, however, that any such reader who expects to find a broad survey of the origin and significance of Greek mathematics will be disappointed ; for this is not so much a straightforward account of the subject as a pleasant ramble with a talkative guide who knows some interesting byways and will entertain us with all kinds of gossip and speculation. The reader may well gain a desire to explore further.

Professor Dantzig begins with a brief introduction on the question, " Who were the Greeks? " This is good and readable, if one is not too irritated by a discursive style, a tendency to dramatisation, and the occasional purple patch. The next chapter, " The Founders ", is on Thales and Pythagoras, chiefly Pythagoras. Professor Dantzig holds, perhaps rightly, that Thales was the greater mathematician of the two, and conjectures that he knew the theorem called by Pythagoras's name, offering however only the slender evidence that Thales is said to have understood similar triangles.

Next we have a chapter on " The Genesis of Geometry ", in which the author works backwards from Euclid to Thales, taking the three unsolved problems on the way. A chapter entitled " Pyramids " is almost entirely devoted to the problem of how Thales measured the height of the Great Pyramid. A conjectured method is put forward, supported by a somewhat

misleading diagram and an explanation in which the terms *ecliptic* and *obliquity of the ecliptic* are wrongly used. The next chapter, " Pentacles ", is concerned with the Golden Section and with regular polygons and polyhedra. He gives an incomplete definition of regular polyhedra, and it is hard to know just what is meant by a statement that " however involved may be the actual construction " (of a regular n-gon) " the difficulty is technical in nature and not a matter of principle ". In this chapter and elsewhere he has much to say about those people, ancient and modern, who love to attribute mystic or occult significance to numbers and geometrical figures. There is next a chapter on " The Pseudomath " (de Morgan's name for the circle-squarers, angle-trisectors, and so on). This is quite entertaining, if a little repetitive, and is enlivened by two examples (more would have been welcome). Part I concludes with a short chapter on the ruler-and-compasses prohibition, treated in general terms and not without some word-spinning.

While it is difficult to believe that any reader will find the first half of the book satisfactory, those who persevere to Part II will find it more meaty and altogether more rewarding. Here Professor Dantzig takes eight particular ideas of Greek mathematics and pursues their development into modern times. The first of these is " The Hypotenuse Theorem ", leading to irrational numbers, analytical geometry and Riemannian ideas ; the next, " Pythagorean Triples ", introducing the theory of numbers ; then " The Crescents of Hippocrates " and " The Quadratrix of Hippias ". He says that the quadratrix was invented " for the express purpose of squaring the circle ", though Heath and Sarton both consider it was more probably for trisecting an angle.

" The Algorithm of Euclid " forms an introduction to continued fractions and " An Archimedean Approximation " leads to interesting related speculations. " The Formula of Hero " is then discussed, but the account given of Hero's method of proof is marred by a mistake on p. 162 and by wrong letters in the figure on the previous page. (There are also mistakes in the figures on pp. 171, 175 in the next chapter.)

" Heronian Triples " lead to a discussion of Pell equations and of Lagrange's work on binomials of the type $x+y\sqrt{R}$. In " The Chords of Hipparchus " he discusses the origin of trigonometry ; but he identifies *chord* with *sine*, whereas Ptolemy's table of chords related the chord to the angle at the centre of the circle, not to that of the circumference.

In a short but interesting " Epilogue " Professor Dantzig considers the limitations of Greek mathematics and looks forward to the next volume of what is to be a trilogy, in which he proposes to discuss the link between the later Greek mathematicians and the revival in the seventeenth century.

E. H. LOCKWOOD

La Théorie du Champ Unifié D'Einstein et quelques-uns de ses Développements. By MARIE-ANTIONETTE TONNELAT. Pp. x + 121. 2,500 fr. 1955. (Gauthier-Villars, Paris)

In or about the year 1949 the daily newspapers published reports of the late Professor Einstein's newest generalized field-theory that, as some of them alleged, would " solve the universe ". Some of the papers published facsimiles of Einstein's new (tensor) field-equations. But the universe has not yet been solved and neither have the equations.

As regards the equations, this is a crude statement. Actually a very great deal of work has been done on them, with Madame Tonnelat as one of the chief investigators. Much is now known about them, including some very particular exact solutions, and some instructive approximate solutions. Probably everything of significance that has so far been discovered is described in this book.

It is well-known that Einstein's proposed unified theory of gravitation and electromagnetism, and, as he hoped, of nuclear forces as well, was based on the replacement of the usual symmetric fundamental tensor g_{pq} by an unsymmetric tensor and of the usual affine connexion determined by the Christoffel symbols by an unsymmetric connexion. E. Schrödinger made very similar proposals at the same time. The corresponding extension of Riemannian geometry and the associated form of tensor calculus will almost certainly be needed sometime for quite other applications. In the long run, Madame Tonnelat's general treatment of the mathematics concerned may prove to be the chief value of her work.

Meantime, she has given an expert unified treatment of the various unified theories that can be developed on this basis. All workers in this field will be grateful to her. But from this point of view most of the book is concerned with highly specialized technicalities and a detailed review would be out of place. This is all the more true because Professor A. Lichnérowicz, with whose flourishing school of French relativists Madame Tonnelat is associated, has contributed a preface that could not be improved upon as a review. But one should close, as he does, by calling attention to the fact that Madame Tonnelat was the first to solve explicitly the equations that relate the affine connexion to the fundamental tensor in such theories. W. H. McCrea

Die Lehre von den Kettenbrüchen. By Oskar Perron. Third Edition. Vol. I: Elementary continued fractions. Pp. vi, 194. DM 29.40. 1954. (Teubner, Stuttgart)

In the first edition of Perron's famous book (published in 1912 and reviewed in Vol. 7 of the *Gazette*) the author was able to assert with truth that no comprehensive exposition of the subject of continued fractions was in existence; it is due to him that the assertion thereupon ceased to be true. Since the day that the book first saw the light it has been not only a fascinating book for the mathematician to read, but also one of the predestined members of the select class of works which are rightly described as mathematical classics.

The book reached a second edition in 1929, a reproduction of the first edition with some very minor changes and additions to bring it up to date; and now, quarter of a century later, we can greet the inspiring friend and companion of some forty years, wearing a new frock but still readily recognisable.

The volume now under review consists of a revised version of the first five chapters, whose subject can be described as elementary continued fractions; the width of the page has been increased by about one-tenth and a somewhat more concise style has been adopted in places. The extent of the additions which have been made may be gathered from the fact that 190 pages of text in the earlier editions have been replaced by 188 pages.

Most of the changes are of quite a minor character; but the book now contains two proofs of Lagrange's theorem on the representation of a quadratic surd by a periodic continued fraction, the older proof by Charves (1877) which was given in previous editions being supplemented by the newer and rather simpler proof by Ballieu (1942). The innate conservatism of the reviewer makes h m feel that the older proof is the more attractive, though age and attractiveness do not always go together (for instance, it is possible to maintain the view that Lagrange's original proof is set out in such a way as to make it almost unreadable). It would, however, have been extremely interesting if the author could have modified the proof due to Charves by introducing simplifications in the manner sketched by Hermite, *Bulletin des sci. math.* IX (1885), p. 11. The existence of this note by Hermite has unfortunately been overlooked by Prof. Perron. (The last two sentences are

inserted merely to satisfy those critics who consider that a reviewer has not done his job unless he has called attention to some error or omission.)

This volume makes the reviewer look forward eagerly to experiencing the thrills which will be given by a sight of the more advanced and more interesting parts of the subject to be contained in the second volume of this new edition.

G. N. WATSON

Die geistige Arbeit. By ANDREAS SPEISER. Pp. 207. 1955. (Birkhäuser, Basle)

The book consists of a collection of lectures and essays ranging over diverse topics in art, science, philosophy and theology. Several of these had previously been published independently. Their underlying thesis is that most important intellectual and artistic productions are intrinsically organised in accordance with some mathematical structure which must be consciously understood for their proper appreciation. Mathematics, science and art form an essential unity, and a strict division between them can be observed only at a loss to each of them. The Renaissance painters, for instance—themselves mathematicians—incorporated in their work the then most modern results in geometry. Art has fallen into decline since its development was outstripped by mathematical progress. The main thesis is developed by analysing a number of ingeniously chosen examples, rather than by a systematic exposition of the author's ideas.

The importance of group theory is stressed, but surprisingly, perhaps, in a book which deals extensively with mathematical thinking, little more than incidental references are to be found regarding foundational research in mathematics later than Fichte. The author claims that mathematics requires a platonistic foundation and that mathematical truth is absolute.

M. H. LÖB

Homological Algebra. By HENRI CARTAN and SAMUEL EILENBERG. Pp. 390. 45s. 1956. (Princeton University Press)

Since the war there have been massive developments in algebraic topology, and certain branches have attained a high degree of independence from the mother subject. The most remarkable of these has now received a monumental exposition in the hands of Henri Cartan and Samuel Eilenberg. This long-awaited book consists of an entirely new mathematical theory, about which little has previously been published. To quote from the Preface :

" During the last decade the methods of algebraic topology have invaded extensively the domain of pure algebra, and initiated a number of internal revolutions. The purpose of this book is to present a unified account of these developments and to lay the foundations of a fully-fledged theory.

" The invasion of algebra has occurred on three fronts through the construction of cohomology theories for groups, Lie algebras, and associative algebras. The three subjects have been given independent but parallel developments. We present herein a single cohomology (and also homology) theory which embodies all three ; each is obtained from it by a suitable specialization.

" This unification possesses all the usual advantages. One proof replaces three. In addition an interplay takes place among the three specializations ; each enriches the other two.

" The unified theory also enjoys a broader sweep. It applies to situations not covered by the specializations. An important example is Hilbert's theorem concerning chains of syzygies in a polynomial ring of n variables. We obtain his result (and various analogous new theorems) as a theorem of homology theory.

" The initial impetus which, in part, led us to these investigations was provided by a problem of topology. Nearly thirty years ago, Künneth studied the relations of the homology groups of a product space to those of the two factors. He obtained results in the form of numerical relations among the Betti numbers and torsion coefficients. The problem was to strengthen these results by stating them in group-invariant form. The first step is to convert this problem into a purely algebraic one concerning the homology groups of the tensor product of two (algebraic) complexes. The solution we shall give involves not only the tensor product of the homology groups of the two complexes, but also a second product called their *torsion* product. The torsion product is a new operation derived from the tensor product. The point of departure was the discovery that the process of deriving the torsion product from the tensor product could be generalized so as to apply to a wide class of functors. In particular, the process could be iterated and thus a sequence of functors could be obtained from a single functor. It was then observed that the resulting sequence possessed the formal properties usually encountered in homology theory."

The work is self-contained, and requires less an extensive knowledge of algebra than an ability to keep in mind the quantities of new functors which the authors keep introducing. There is an absence of external motivation throughout, and the topological background is hardly mentioned at all. The work is to be appreciated first of all as a piece of pure, creative algebra, but towards the end of the book there are chapters on the applications. These refer to finite groups, with class-field theory especially in mind, Lie algebras, and the classification of extensions for groups, algebras, and modules. There is a useful exposition of spectral sequences, which is somewhat independent of the rest of the book. In an appendix, which might very well be read before the book itself is begun, David Buchsbaum places the whole work in a more general setting with his theory of exact categories. There is a good index, and an invaluable list of the hundred-odd symbols which Eilenberg and Cartan use with a special meaning.

I. M. JAMES

Mechanics. By A. H. G. PALMER and K. S. SNELL. Second Edition. Pp. xvi, 352. 25s. 1956. (University of London Press)

With the cost of producing a new textbook at its present high figure, a fresh edition of a pre-war publication has a certain presumption in its favour. If in addition there has been no marked change in fashion, touched off perhaps by the Mathematical Association, which would too clearly date the work ; then it stands a good chance of retaining a measure of popularity.

This edition differs from the first only in the addition of some new material : a set of easier examples to follow the vector introduction, and an appendix on scalar and vector products, which is well arranged.

The weakness of the work, now as in 1935, lies in the assumption that after some " mechanics before the school certificate . . . usually tackled in as practical way as possible " the pupil's best path to the coherent theory of the subject is *via* the abstract theory of vectors. Most teachers prefer to develop the theory of mechanics in its own right, with the aid of all its concrete associations ; believing that vector theory, especially in its analytical form, is understood through statics rather than vice versa.

One topic in particular which is poorly treated is relative velocity. The preliminary course of practical mechanics must deal very thoroughly with this if the highly condensed and abstract discussion is to do more than confuse.

However, the authors by the time they reach Chapter V have shed their excessive preoccupation with vector theory, and from this point onwards we have a workmanlike " second course " leading to A-level and scholarship, with

adequate worked examples well set out and plenty of practice examples. The rigid dynamics section in particular should be commended for consistently using two diagrams, one for force and one for kineta; or one for impulses and one for changes in linear and angular momenta. The engineers and perhaps some of the physicists will insist on their D'Alembert, but it is good to find an A-level applied mathematics from which he is firmly excluded.

No doubt many teachers will find this a useful text for the mathematical specialist, even if it is not ideal. For the mechanics teacher above all, no book is ideal except the one he hopes to write. A. J. MOAKES

Nichtlineare Regelungsvorgänge. Edited by W. HAHN. Pp. 107. 1956. (Oldenbourg, Munich)

Five memoirs on problems of control systems which involve non-linear equations are collected here. Magnus writes on the conflation of the "harmonic balance" process of linear approximation with the Routh-Hurwitz criterion for stability, Janssen and Vermeulen on problems involving friction and hysteresis, Hahn on a development of one of the Liapounoff stability tests, Pestel on a problem concerning guided weapons, and Haacke on the phase-plane metnod applied to a feed-back system with relays. The collection should be valuable to those already familiar with the elements of the theory of control. T. A. A. BROADBENT

Numerical Mathematical Analysis. By J. B. SCARBOROUGH. Third Edition. 48s. 1955. (Baltimore, John Hopkins Press; London, Oxford University Press)

It is a quarter of a century since the first appearance of this work. The present reviewer, then a young man with a moderate experience of teaching numerical analysis, had occasion then to review the work, finding much to praise and a few things, notably the very cumbersome treatment of central difference formulae of interpolation and the old-fashioned treatment of the normal law of error and the principle of least squares, which seemed to him then to be capable of much improvement. This review appeared in *Math. Gaz.* XVI (1932), p. 61.

We have just read this old review with great care, and must voice extreme disappointment in finding that the criticisms still hold for those particular sections of the third edition now being reviewed. It is necessary, for the sake of those who may possibly use the book, to state once and for all that the theoretical treatment of the central difference formulae of interpolation, namely the formulae of Stirling and of Bessel, deriving of course from Newton's general formula (never mentioned in this book), is quite unnecessarily cumbersome. On the other hand, the sections on solving simultaneous equations, and ordinary and partial differential equations, also integral equations, seem well done and very informative.

Some casual addenda. The worked Example 3 on p. 456 is wrong, and very misleading. If data, themselves forced to the nearest integer of some scale-reading, are hopelessly ill-conditioned, no amount of merely *arithmetical* accuracy (as here in taking logarithms to more places of decimals) will cure the intrinsic *statistical* inaccuracy, and to suggest that it may, by the *pessimum exemplum* of a "Perfunctory Computation" corrected (this should also be in inverted commas) by a more arithmetically accurate computation, is not conducive to clear ideas. In a word, the question is not only of numerical accuracy but also of statistical significance. However, it must be admitted that the author might claim certain august company in this kind of error.

There are some misprints, very few; *e.g.* "mainily" for "mainly" in line 7 of the Preface of the Third Edition, p. 6. A. C. AITKEN

Elementary Wave Mechanics with Applications to Quantum Chemistry. 2nd Edition. By W. HEITLER. Pp. viii, 193. 18s. 1956. (Oxford: Clarendon Press. London: Cumberlege)

This book, which the preface describes as being " . . . primarily designed for the use of chemists and other non-mathematical readers . . .", differs from the first edition in having a chapter on diatomic molecules and a more detailed treatment of the chemical bond. Both these additions increase the value of the book to chemists, although a greater use of pictorial representation of covalent bonds would be welcome.

In the first half of the book there is an introductory account of the experimental basis of quantum mechanics and of the conception of uncertainty, followed by chapters on the wave equation and its application to the hydrogen atom, the Zeeman effect and perturbation theory. The second half is devoted to subjects of particular interest to chemists. There are chapters on the application of quantum mechanics to the periodic system of elements, the theory of the homopolar chemical bond and to valency. The standard of the mathematics required for a full understanding of the text is mainly only Advanced Level, and Professor Heitler has gone to great pains to explain the physical significance of all the mathematical operations used.

The book is written in a very readable style and is to be highly recommended to all students, particularly of chemistry, who are starting a study of wave mechanics.

E. R. A. PEELING

Spheroidal Wave Functions including Tables of Separation Constants and Coefficients. By J. A. STRATTON, P. M. MORSE, L. J. CHU, J. D. C. LITTLE and F. J. CORBATÓ. Pp. xiii, 613. 100s. 1956. (Published jointly by the Technology Press of M.I.T. and John Wiley, New York; Chapman & Hall, London)

This weighty volume enshrines the results of extensive computations along lines which, though considerably changed in detail, are the same in general purpose as those already familiar from the much smaller work entitled *Elliptic Cylinder and Spheroidal Wave Functions* put forth by Stratton, Morse, Chu and Hutner in 1941; this is indeed referred to in Morse's Foreword as " the previous edition ". In both, the core of the explanatory text is a reprint of a paper by Chu and Stratton (Elliptic and spheroidal wave functions, *J. of Math. and Phys.*, **20**, 259–309, 1941). In replacement of the previous introduction by Mrs. Hutner, the present work contains (pages 53–62) an " Introduction to the Tables " by Little and Corbató. This new introduction establishes the notation and normalization used by Morse and Feshbach in their *Methods of Theoretical Physics*; it then compares these with the formulation of Chu and Stratton, and describes the tables, which are now more nearly on the system of Morse and Feshbach; it concludes by describing the programming for the calculation of the tables on the Whirlwind I computer.

Whereas the earlier work contained, as far as spheroidal wave functions are concerned, a mere 42 pages of tables, the present volume contains 540 pages of tables, namely four 135-page tables giving coefficients for prolate radial, prolate angular, oblate radial and oblate angular functions, all of the first kind. The earlier work tabulated coefficients for angular functions only, both prolate and oblate, but of both the first and the second kinds. Both publications also include information relating to separation constants. Owing to changes of system, numerical values in the two works are mostly not particularly readily comparable.

In the case of the radial functions, the coefficients are those in expansions in terms of spherical Bessel functions; in the case of the angular functions,

the coefficients are those in expansions in terms of associated Legendre functions. According to the Foreword, " it is expected that eventually a larger volume will be published which will give actual numerical values of the solutions themselves ".

To quote again from Morse's Foreword, " the present tables were computed, tabulated, *and printed* automatically ; no human intervention entered between the introduction of the program and the typed page, ready for offset photography ". In order to remove any impression that automation necessarily implies a table-maker's paradise, one may quote also from Little and Corbató : " The accuracy of the tables has been checked by the hand calculation of a number of cases. For the most part, all the significant figures published are good except for an occasional error of one in the last place. When the coefficient itself gets large ($\sim 10^2$), the last place may be weak. At no time is it thought that the numbers are off more than 5 in the last place." It seems that these 7-figure tables should always be reliable to 6 figures.

References are given to work by other authors. The outstanding impression, however, is of the monumental contribution made by the M.I.T. group. Anyone working on computations involving spherical wave functions, say in acoustics, radar or quantum theory, will surely be grateful that such extensive tables are now available.

A. FLETCHER

Tutorial Algebra. Vol. II. By W. BRIGGS and G. H. BRYAN. Sixth Ed. Revised and enlarged by G. WALKER. Pp. v + 725. 24s. 1956. (University Tutorial Press)

This volume is intended for candidates for London B.A. and B.Sc. General Examination and University Entrance Scholarship candidates. The first third of the book deals with analysis, convergence of series, continuity and the elementary functions. This is followed by chapters on complex numbers, finite differences, summation of series, determinants, matrices, elimination and the theory of equations. The book contains an immense number of worked examples, and at the price is very good value indeed for the student in question.

The chapter on matrices, written by Dr. A. Mary Tropper, is exceptionally good and provides a really first rate, easy to read introduction. The chapter on finite differences is also very good and interesting. Throughout the volume there is evidence of great care having been taken with definitions, but the quality of the book is uneven, and there is some want of coordination between the chapters ; for instance, on p. 487 there is a long-winded verification of the solution of the equation $u_{n+1} - au_n + u_{n-1} = 0$ which had already been well treated in a previous chapter. The following specific points may be noted. On p. 18 the proof of Cauchy's convergence principle is completely false because M, m here depend upon ϵ. On p. 23 the use of s_p for the sum to p terms of one series and s_n for the sum to n terms of another, in the same proof, is hopelessly confusing. The proof on p. 24 about the sum of two absolutely convergent series is confusing because the convergence of $\Sigma(u_r + v_r)$ to $\Sigma u_r + \Sigma v_r$ is independent of absolute convergence. It is unsatisfactory to say (p. 38) that if $\Sigma a_n x^n$ converges, then it is easy to see that $\Sigma a_n x^n \to a_0$ as $x \to 0$; this is no easier than other steps proved in detail and merits careful discussion. In § 1.81, p. 43, the exception $f(x) = 0$ identically should be excluded. The proof of § 2.64, p. 75, fails if $\phi(x) = \alpha$ for $0 < |x - a| < n_1$. On p. 87, the proof of (ii) is unnecessarily complicated and unconvincing ; given $\epsilon > 0$, for $x \neq 0$, $|nx/(1 + n^2x^2)| < 1/n |x| < \epsilon$ for $n > 1/|x| \epsilon$ so that

$$|nx/(1 + n^2x^2)| < \epsilon$$

if $x = 0$ or if $x \neq 0$ and $n > 1/|x| \epsilon$, but for $x = 1/n$, $nx/(1 + n^r x^r) = 1/2$ which shows

that the convergence is not uniform. In the proof of the Binomial Theorem for a rational index, we meet the familiar argument due to Euler that since

$$\Sigma\binom{n}{r-p}\binom{m}{p}-\binom{m+n}{r}$$

is of the rth degree in both m and n and vanishes for all integral m, n, therefore it vanishes for *all* m, n. This is by no means trivial as the following proof may show. Let $P(x, y)$ be the polynomial

$$x^n a_n(y)+x^{n-1}a_{n-1}(y)+\ldots+xa_1(y)+a_0(y) \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

where each $a_r(y)$ is a polynomial in y. We have to prove that if $P(x, y)=0$ for all positive integers x, y, then $P(x, y)=0$ identically. Let y be a fixed positive integer. Then (i) vanishes for all positive integral values of x, and since it is a polynomial of the nth degree in x, the coefficients of the polynomial are all zero. Thus $a_r(y)=0$, $r=0, 1, 2, \ldots, n$ and this is true for any integer y; but each $a_r(y)$ is a polynomial and so each vanishes identically, which proves that $P(x, y)$ vanishes identically.

The proof of Abel's theorem (p. 154) is inadequate; on the results established in the book the uniform convergence of $\Sigma a_n x^n$ is assured only in any closed interval contained in $(-1, +1)$, not necessarily in an interval containing the point $x=1$. On p. 160 the inequality for the remainder in the exponential series $|R_n|<|x|^n/n!$ if $x<0$, may be invalid if $n+1<x$. The conditions for divergence in Kummer's test (p. 245) is wrongly given as

$$u_n d_n/u_{n+1}-d_{n+1}\leqslant l<0 \text{ for } n\geqslant q,$$

whereas in fact we may replace l by zero.

The proof (p. 286) of the triangle inequalities $|z_1+z_2|\leqslant|z_1|+|z_2|$, $|z_1-z_2|\geqslant||z_1|-|z_2||$ by repeated squaring is clumsy and outdated, and the second is of course an immediate consequence of the first. The best proof uses $|z_1+z_2|^2=(z_1+z_2)(\bar{z}_1+\bar{z}_2)$ and $w+\bar{w}=2Rlw<2|w|$ unless w is real and positive (taking $w=z_1\bar{z}_2$), which yields the condition for equality that z_1/z_2 be real and positive. On p. 315 the argument seems to be that if $R(z)\to 0$ then $R(z)/z\to 0$ as $z\to 0$. In the discussion of the Gauss test (p. 320) for complex series, the reader is referred to Bromwich's infinite series for a proof of convergence in the case of $0<R(\mu)\leqslant 1$, $(z\neq 1)$, but in fact Bromwich does not himself give a proof of this case, just a reference to an article by Pringsheim. The treatment of partial fractions is clumsy and there is no reference to Hospital's rule even though many of the examples discussed are designed to be applications of the rule (for instance, Example 4, p. 357). To find the partial fractions of $x/(x-1)(2x-1)(x-2)^n$, the long discussion on p. 359 is unnecessary; writing $x-2=y$ the given expression becomes

$$\left\{\frac{1}{y+1}-\frac{1}{2y+3}\right\}\frac{1}{y^n}$$

and since

$$\frac{1}{1+y}=1-y+y^2+\ldots+(-y)^{n-1}+\frac{(-y)^n}{1+y}$$

with a similar expression for $\dfrac{1}{2y+3}$ the fractions may be written down immediately.

R. L. GOODSTEIN

Gewöhnliche differentialgleichungen. By G. HOHEISEL. Fifth Ed. Sammlung Göschen 920. Pp. 129. DM 2.40. 1956. (De Gruyter, Berlin)

As one expects from a book in this collection an immense amount of information is packed into a small space without sacrificing accuracy or compre-

hensibility. There are no exercises, but theory is amply illustrated by worked examples. The existence and uniqueness of the solution of $dy/dx = f(x, y)$ under a Lipschitz condition is established and the methods of solution of the familiar integrable types are described. In addition there are accounts of Green's function, and eigen-functions of $L(y) + \lambda g(x)y$. The final section deals with the zeros of the solutions of $y'' + Q(x)y = 0$, proving for example that the zeros of linearly independent solutions are interlacing. R.L.G.

An Essay on the Foundations of Geometry. By BERTRAND A. W. RUSSELL. Pp. 201. $1.50. 1956. (Dover Publications, New York)

This is an unaltered and unabridged reprint of Russell's famous Fellowship dissertation of 1897. Much that Russell sought to establish sixty years ago is now a commonplace of every undergraduate course, but every topic is illuminated by Russell's genius, his power to unravel a complicated argument and to hit the nail on the head with a few deft strokes. In the introduction written for this edition Morris Kline claims that the theory of relativity refutes Russell's contention that motion of rigid bodies is possible only in homogeneous space (space of constant curvature). Russell says (p. 75): "It follows that measurements cannot, without a vicious circle, be itself derived from experience of rigid bodies. Geometrical measurement, in fact, is the comparison of spatial magnitudes, and such comparison involves ... the homogeneity of space. ... Without the homogeneity of space the very notion of rigidity or non-rigidity could not exist." The theory of relativity does not refute Russell's analysis of motion and measurement, but escapes Russell's conclusion *by changing the meaning of measurement*. On the independence of geometry and spatial experience Russell finds in perspective geometry the fulfilment of Grassmann's vision of a Geometry wholly a creature of the intellect yet dealing with extension, as a concept not a percept. By contrast he says of metrical geometry (p. 147): "the axioms peculiar to Euclid—which are properly not axioms but empirical results of measurement—determine within the errors of observation which of these *a priori* possibilities (the Euclidean and the non-Euclidean geometries) is realised in our actual space." R. L. GOODSTEIN

Der vierdimensionale Raum. By R. W. WEITZENBÖCK. Pp. 224. Fr/DM 19.55. 1956. Sammlung Wissenschaft und Kultur, Bd. 10. (Birkhäuser, Basel)

This is an amplification and revision of a book with the same title which the author published in 1929. It should be clearly understood that this is not a text-book on geometry of 4-dimensions or on relativity theory, but a systematic account of different ways in which the fourth dimension plays a part in the various branches of knowledge.

After an introductory chapter, the reader is invited to consider space of 4-dimensions as a geometer's fairyland in the manner of Abbot's "Flatland". The third chapter deals with special and general relativity mainly from a philosophical point of view, the space of 4-dimensions being interpreted as the space-time continuum. The fourth chapter surveys the relation between 4-dimensional space and various branches of knowledge, including physics, chemistry, astronomy, religion and metaphysics. The book is completed by a chapter dealing with the part that 4-dimensional space has played in literature, including science fiction.

In many respects the book is unusual for, although a serious and scholarly work, it includes in its bibliography such sources as Ellery Queen's *The Siamese Twin Mystery*, *The Time Machine* by H. G. Wells, and *Laughing Gas* by P. G. Wodehouse. Interesting to the specialist scientist, it can also be read with profit by the thoughtful layman. T. J. WILLMORE

From Zero to Infinity. By CONSTANCE REID. Pp. 145. 12s. 6d. 1956. (Routledge & Kegan Paul)

This is primarily an account for the general reader of some problems in elementary number theory in their historical setting. The story is well told and should interest a wide public. Shaw's black girl's account of the integers could hardly be improved upon, but on such topics as Mersenne numbers and the law of quadratic reciprocity *From Zero to Infinity* is lucid, sound and entertaining.

R. L. GOODSTEIN

Mathematische Formelsammlung. By F. O. RINGLEB. Sammlung Göschen 51/51a. Pp. 278. DM 4,80. 1956. (de Gruyter, Berlin)

This collection of formulae covers a very wide range of topics: algebra, number theory, plane and spherical trigonometry, coordinate and differential geometry in two and three dimensions, a little field theory, elementary real variable function theory, the complex integral and differential equations. The material is clearly indexed and easy to find, the printing bold and clear; it should make a very useful reference book for students preparing for a Degree Examination. It would be preferable in the next edition to give the rule for multiplying determinants in the matrix multiplication form of rows by columns rather than in the form of rows by rows, to give the value of the integral $\int (1/x)\,dx$ as $\log |cx|$ and not $\log cx$ and to give the rule for substitution in the definite integral under less restrictive conditions.

R. L. GOODSTEIN

Colloque sur la Théorie des Nombres. Bruxelles 1955. (Georges Thone, Liège; Masson & Cie, Paris)

The first part of this volume consists of a reprint of the celebrated paper by M. Ch.-J. de la Vallée Poussin (Louvain), "Sur la fonction $\zeta(s)$ de Riemann et le nombre des nombres premiers inférieurs à une limite donnée"—Extrait des Mémoires in-8° de l'Académie royale de Belgique, Classe des Sciences, 1898, tome LIX. In 1896 both Hadamard and de la Vallée Poussin proved independently the long conjectured prime number theorem

$$\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1,$$

$\pi(x)$ denoting the number of primes not exceeding x. In the reprinted paper of 1898 the first estimate of an error term was given, namely, the author proved the result

$$\left|\pi(x) - \int_0^x \frac{du}{\log u}\right| < ax\, e^{-b\sqrt{\log x}}, \qquad (1)$$

where a and b are absolute positive constants and the value of b is explicitly given. In addition, de la Vallée Poussin proved that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0,$$

where $\mu(n)$ is the Möbius function. All these problems are closely connected with the problem of the distribution of the function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

and (1) is essentially equivalent to the theorem that for all complex zeros $\sigma_0 + t_0 i$ of $\zeta(s)$ the expression

$$(1-\sigma_0)(|\log|t_0||+$$

has a positive lower bound. The mathematical world owes le Centre Belge de Recherches Mathématiques a great debt for having made this paper available to a wide circle of readers.

The second part of the publication consists of a number of papers given at a Colloquium on the theory of numbers in Brussels in December 1955. Of special interest is the paper by F. K. Roth, who proves that for each real algebraic number α and each $\epsilon > 0$ the diophantine inequality

$$|y\alpha - x| < y^{-1-\epsilon} \quad \ldots\ldots\ldots\

between systems, information), and Regulation and Control (the central theme of cybernetics). The stress is on a functional and behaviouristic attitude and it is stated that " The truths of cybernetics are not conditional on their being derived from some other branch of science " (p. 1). Of course, terms are being borrowed from other sciences, such as transformation, or Markov chains. An idea is given of Shannon's theory of communication and of his fundamental theorem about transmission of information in the presence of noise. " The reader who finds this incredible must go to Shannon's book for the proof " (p. 190).

After a study of the book the reader will have obtained a good knowledge of what cybernetics is about. By working through the examples, he will also have acquired some skill in producing those answers which the author expects. However, one cannot help wondering whether the lasting impression will not be that of having been introduced to trivialities. It is true that the reader will have learned how to express some commonsense knowledge in precisely worded terms, but is this sufficient reward for his attention? To a mathematician, at any rate, the deliberate dodging of mathematics takes the fundamentals out of an intriguing subject. One misses also the statistical flavour of Wiener's book on the same subject, and there is no mention of social implications.

Even so, one enjoys reading the book for its very personal style. The examples given in the text as well as in the exercises for the reader (with solutions) are varied and amusing. On p. 169 the hint given after one example is " Beware ", and a useful hint it is. On p. 230 the author explains why he prefers the phrase " hunt and stick " to the more usual " trial and error ", and a unit of large-scale selection is called a Megapick on p. 261.

The reviewer would think that the author's attempts at finding telling relationship leads him occasionally astray. His reference to the theory of games (p. 241) is superficial and misleading. No connection is apparent between the central theorem of games theory, *i.e.* the Minimax Theorem, and cybernetics.

The word Introduction in the title should be taken more seriously than with some other books, but we may assume that biologists can learn a great deal here which they could not get equally conveniently from other sources. Editing was done with great care, and paper and print are excellent.

S. VAJDA

Oxford Graded Arithmetic Practice. Book Five : **Addition and subtraction of Money.** Book Six : **Multiplication and Division of Money.** By D. A. HOLLAND. Pp. 64. 2s. 6d. With answers, 3s. 6d. 1955. (Oxford University Press)

The teacher who is looking for mechanical exercises in these processes will find his need richly filled in these books. The steps leading to addition of £.s.d.f. have been carefully analysed and spread over 13 stages. In the teacher's book each step is described at the top of the page and indexed on page 2. By working horizontally instead of vertically the exercises may be used for diagnosis of difficulties. The first five pages give examples of change of unit and of examples to be worked in the head only. The other three processes are treated similarly.

The lay-out of these books is extremely clear and the figures are large. In the teacher's book the answers are given in position and in red—a most useful arrangement for this type of book.

The mental age range for each book is 4 years (7–11 or 8–12), and one wonders how their use would be organised in a school where each of four classes wished to use a different section.

K. SOWDEN

Calculus Refresher for Technical Men. By A. A. KLAF. Pp. 431. $1.95. 1956. (Dover Publications, New York)

Trigonometry Refresher for Technical Men. By A. A. KLAF. Pp. 629. $1.95. 1956. (Dover Publications, New York)

These collections of formulae, methods, worked examples and exercises cover a wide field. The applications of the calculus, for instance, include not only arc-length, area, volume and mean-values, but also statics, hydrostatics, electricity and kinematics. The Trigonometry volume includes both plane and spherical trigonometry, with applications to sea and air navigation, complex numbers and series. As collections of worked examples and formulae the two volumes have their merits, but most of the attempts at definition, exposition of fundamental ideas and proof are quite pitiful. For example, in the Calculus book (p. 2), *function* and *relation* are confused when the equation $x^2+y^2=r^2$ is called a function of x and y; the definition of limit (p. 7) is meaningless (instead of defining $\lim f(x)$ as $x \to a$, we find instead the old bogey of the variable with a limit, and (p. 23), its offspring, the infinitesimal); if $a_n(r) \to a(r)$ it is assumed (p. 115) that $\Sigma a_n(r) \to \Sigma a(r)$; having proved (assuming without comment the differentiability of a power series) that the derivative of e^x is e^x, the derivative of e^{ax} is then (p. 118) sought for by expanding $(1+1/n)^{nax}$; we are told (p. 171) that nothing can be integrated before the inverse process of differentiation has produced that which must be integrated; and (on p. 385) the binomial theorem

$$(a+b)^n = a^n + \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \ldots$$

is said to be true when n is integral or fractional.

The Trigonometry volume, on the other hand, is much less open to criticism in this respect. The treatment of inverse functions, for instance, is quite good. The derivation of the expansions for $\sin x$ and $\cos x$ (p. 428) is however, unsound, and it cannot be of much help to be told (p. 394) that "an imaginary number in comparison with a whole number is, *generally speaking*, no more unreal than is a fraction or an irrational number" (my italics).

R. L. GOODSTEIN

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